Exercises in Introduction to Mathematical Statistics (Ch. 2)

Tomoki Okuno

September 14, 2022

Note

- Not all solutions are provided: exercises that are too simple or not very important to me are skipped.
- Texts in red are just attentions to me. Please ignore them.

2 Multivariate Distributions

2.1 Distributions of Two Random Variables

2.1.1. Let \( f(x_1, x_2) = 4x_1x_2, \) \( 0 < x_1 < 1, \) \( 0 < x_2 < 1, \) zero elsewhere, be the pdf of \( X_1 \) and \( X_2. \) Find \( P(0 < X_1 < \frac{1}{2}, \frac{1}{4} < X_2 < 1), \) \( P(X_1 = X_2), \) \( P(X_1 < X_2), \) and \( P(X_1 \leq X_2). \)

Solution.

\[
P \left( 0 < X_1 < \frac{1}{2}, \frac{1}{4} < X_2 < 1 \right) = \int_{1/4}^{1/2} \int_{X_1}^{1/2} 4x_1x_2dx_1dx_2 = \cdots = \frac{15}{64}
\]

\( P(X_1 = X_2) = 0 \) since the support is a segment not area

\[
P(X_1 < X_2) = \int_{0}^{1} \int_{0}^{x_2} 4x_1x_2dx_1dx_2 = \int_{0}^{1} 2x_1^2x_2|_{x_1=0}^{x_2}dx_1dx_2 = \int_{0}^{1} 2x_2^3dx_2 = \frac{1}{2}.
\]

\[
P(X_1 \leq X_2) = P(X_1 < X_2) + P(X_1 = X_2) = P(X_1 < X_2) = \frac{1}{2}.
\]

2.1.2. Let \( A_1 = \{ (x, y) : x \leq 2, y \leq 4 \}, \) \( A_2 = \{ (x, y) : x \leq 2, y \leq 1 \}, \) \( A_3 = \{ (x, y) : x \leq 0, y \leq 4 \}, \) and \( A_4 = \{ (x, y) : x \leq 0, y \leq 1 \} \) be subsets of the space \( A \) of two random variables \( X \) and \( Y, \) which is the entire two-dimensional plane. If \( P(A_1) = \frac{7}{8}, \) \( P(A_2) = \frac{4}{8}, \) \( P(A_3) = \frac{3}{8}, \) and \( P(A_4) = \frac{2}{8}, \) find \( P(A_5), \) where \( A_5 = \{ (x, y) : 0 < x \leq 2, 1 < y \leq 4 \}. \)

Solution. \( P(A_5) = P(A_1) - P(A_2) - P(A_3) + P(A_4) = \frac{2}{8}. \)

2.1.3. Let \( F(x, y) \) be the distribution function of \( X \) and \( Y. \) For all real constants \( a < b, c < d, \) show that

\[
P(a < X \leq b, c < Y \leq d) = F(b, d) - F(b, c) - F(a, d) + F(a, c).
\]

Solution.

\[
P(a < X \leq b, c < Y \leq d) = P(X \leq b, c < Y \leq d) - P(X \leq a, c < Y \leq d)
\]

\[
= P(X \leq b, Y \leq d) - P(X \leq b, Y \leq c) - P(X \leq a, Y \leq d) + P(X \leq a, Y \leq c)
\]

\[
= F(b, d) - F(b, c) - F(a, d) + F(a, c).
\]

2.1.7. Let \( f(x, y) = e^{-x-y}, \) \( 0 < x < \infty, \) \( 0 < y < \infty, \) zero elsewhere, be the pdf of \( X \) and \( Y. \) Then if \( Z = X + Y, \) compute \( P(Z \leq 0), \) \( P(Z \leq 6), \) and, more generally, \( P(Z \leq z), \) for \( 0 < z < \infty. \) What is the pdf of \( Z. \)
Solution.

Compute the general probability:

\[ F(z) = P(Z \leq z) = P(X + Y \leq z) = P(Y \leq -X + z) = \int_0^z \int_0^{z-x} e^{-x-y}dydx = \int_0^z (e^{-x} - e^{-z})dx = 1 - e^{-z} - ze^{-z}. \]

Hence, \( P(Z \leq 0) = 0, \) \( P(Z \leq 6) = 1 - 7e^{-6}, \) and \( f(z) = F'(z) = ze^{-z}, \) \( 0 < z < \infty, \) zero elsewhere.

2.1.8. Let \( X \) and \( Y \) have the pdf \( f(x, y) = 1, \) \( 0 < x < 1, \) \( 0 < y < 1, \) zero elsewhere. Find the cdf and pdf of the product \( Z = XY. \)

Solution.

If \( z \leq 0, \) then \( F(z) = P(Z \leq z) = 0 \) because \( Z > 0. \)

\[ F(z) = P(Z \leq z) = P(Y \leq z/X) = \int_0^z \int_0^{1/y} dydx + \int_z^1 \int_{X(z)}^{z/y} dydx = z - z \log z, \quad 0 < z < 1, \]

and one \( z \geq 1. \) Hence, the pdf pf \( Z \) is

\[ f_Z(z) = F'(z) = -\log z, \quad 0 < z < 1, \]

to zero elsewhere.

2.1.11. Let \( X_1 \) and \( X_2 \) have the joint pdf \( f(x_1, x_2) = 15x_1^2x_2, \) \( 0 < x_1 < x_2 < 1, \) zero elsewhere. Find the marginal pdfs and compute \( P(X_1 + X_2 \leq 1). \)

Solution.

\[ f_{X_1}(x_1) = \int_{x_1}^{1} 15x_1^2x_2dx_2 = \frac{15x_1^2(1 - x_1^2)}{2}, \quad 0 < x_1 < 1, \]

\[ f_{X_2}(x_2) = \int_{0}^{x_2} 15x_1^2x_2dx_1 = 5x_2^4, \quad 0 < x_2 < 1, \]

\[ P(X_1 + X_2 \leq 1) = 15 \int_{0}^{1/2} x_1^{1/2} \left( \int_{x_1}^{1-x_1} x_2dx_2 \right) dx_1 = \cdots = \frac{5}{64}. \]

2.1.13. Let \( X_1, X_2 \) be two random variables with the joint pmf \( p(x_1, x_2) = (x_1 + x_2)/12, \) for \( x_1 = 1, 2, x_2 = 1, 2, \) zero elsewhere. Compute \( E(X_1), E(X_1^2), E(X_2), E(X_2^2), \) and \( E(X_1X_2). \) Is \( E(X_1X_2) = E(X_1)E(X_2)? \) Find \( E(2X_1 - 6X_2^2 + 7X_1X_2). \)

Solution.

First, find the marginal pdfs:

\[ p_{X_1}(x_1) = \sum_{x_2=1}^{2} \frac{x_1 + x_2}{12} = \frac{x_1 + 1}{12} + \frac{x_1 + 2}{12} = \frac{2x_1 + 3}{12}, \quad p_{X_2}(x_2) = \frac{2x_2 + 3}{12}. \]

Hence

\[ E(X_1) = \sum_{x_1=1}^{2} x_1p(x_1) = p_{X_1}(1) + 2p_{X_1}(2) = \frac{5}{12} + \frac{14}{12} = \frac{19}{12}; \]

\[ E(X_1^2) = p_{X_1}(1) + 2^2p_{X_1}(2) = \frac{33}{12}; \]

\[ E(X_2) = E(X_1) = \frac{19}{12}, \quad E(X_2^2) = E(X_2^2) = \frac{33}{12}. \]
Also, use the joint mgf to obtain

\[ E(X_1X_2) = \sum_{x_1,x_2} x_1x_2 p(x_1,x_2) = p(1,1) + 2p(2,1) + 2p(1,2) + 4p(2,2) = \frac{5}{2} \neq E(X_1)E(X_2). \]

Therefore,

\[ E(2X_1 - 6X_2^2 + 7X_1X_2) = \frac{19}{12} - 6 \frac{33}{12} + 7 \frac{5}{2} = \frac{25}{6}. \]

2.1.15. Let \( X_1, X_2 \) be two random variables with joint pmf \( p(x_1,x_2) = (1/2)^{x_1+x_2} \), for \( 1 \leq x_i < \infty \), \( i = 1,2 \), where \( X_1 \) and \( X_2 \) are integers, zero elsewhere. Determine the joint mgf of \( X_1, X_2 \). Show that \( M(t_1,t_2) = M(t_1,0)M(0,t_2) \).

\[ p(x_1) = \sum_{x_2=1}^{\infty} \frac{1}{(2)^{x_1+x_2}} = \frac{1}{1-\frac{1}{2}} = \frac{1}{2}^{x_1}, \quad p(x_1) = \frac{1}{2}^{x_2} \]

\[ M_{X_1}(t) = \sum_{x_1=1}^{\infty} \left( \frac{e^t}{2} \right)^{x_1} = \frac{e^t/2}{1-e^t/2} = \frac{e^t}{2-e^t} = M_{X_2}(t), \quad t < \log 2, \]

\[ M(t_1,t_2) = \sum_{x_1=1}^{\infty} \sum_{x_2=1}^{\infty} \left( \frac{e^{t_1/2}}{2} \right)^{x_1} = \frac{1}{2}^{x_1} \frac{1}{2}^{x_2} = \sum_{x_1=1}^{\infty} \frac{e^{t_1/2}}{2} \sum_{x_2=1}^{\infty} \frac{e^{t_2/2}}{2} \]

\[ = M_{X_1}(t_1)M_{X_2}(t_2) = M(t_1,0)M(0,t_2). \]

2.2 Transformations: Bivariate Random Variables

2.2.1. If \( p(x_1,x_2) = \left( \frac{2}{3} \right)^{x_1+x_2} \left( \frac{1}{3} \right)^{2-x_1-x_2} \), \( (x_1,x_2) = (0,0), (0,1), (1,0), (1,1) \), zero elsewhere, is the joint pmf of \( X_1 \) and \( X_2 \), find the joint pmf of \( Y_1 = X_1 - X_2 \) and \( Y_2 = X_1 + X_2 \).

\[ p_{Y_1,Y_2}(y_1,y_2) = p \left( \frac{y_1 + y_2}{2}, \frac{y_2 - y_1}{2} \right) = \left( \frac{2}{3} \right)^{y_1} \left( \frac{1}{3} \right)^{2-y_1} \]

zero outside the support.

2.2.5. Let \( X_1 \) and \( X_2 \) be continuous random variables with the joint pdf \( f_{X_1,X_2}(x_1,x_2) \), \( -\infty < x_i < \infty \), \( i = 1,2 \). Let \( Y_1 = X_1 + X_2 \) and \( Y_2 = X_2 \).

(a) Find the joint pdf \( f_{Y_1,Y_2} \).

\[ f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(y_1 - y_2, y_2) | J | = f_{X_1,X_2}(y_1 - y_2, y_2). \]

(b) Show that

\[ f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(y_1 - y_2, y_2) dy_2, \quad (2.2.5) \]

which is sometimes called the convolution formula.

\[ \text{Solution.} \]

The support is \( -\infty < y_1 - y_2 < \infty, -\infty < y_2 < \infty, \text{i.e., } -\infty < y_i < \infty, \ i = 1,2 \), which gives (2.2.5).
2.2.6. Suppose \( X_1 \) and \( X_2 \) have the joint pdf \( f(x_1, x_2) = e^{-(x_1+x_2)}, 0 < x_i < \infty, i = 1, 2, \) zero elsewhere.

(a) Use formula (2.2.5) to find the pdf of \( Y_1 = X_1 + X_2. \)

Solution.

Since the support of \( Y \) is \( 0 < y_1 - y_2 < \infty, 0 < y_2 < \infty \Rightarrow 0 < y_2 < y_1 < \infty, \)

\[ f_Y(y_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(y_1 - y_2, y_2)dy_2 = \int_{y_2/2}^{y_1} e^{-y_1}dy_2 = y_1e^{-y_1}, \quad y_1 > 0. \]

(b) Find the mgf of \( Y_1 \)

Solution.

\[ M(t) = \int_{0}^{\infty} y_1e^{-(1-t)y_1}dy_1 = \Gamma(2) \left( \frac{1}{1-t} \right)^2 = \frac{1}{(1-t)^2}, \quad t < 1. \]

2.2.7. Use the formula (2.2.5) to find the pdf of \( Y_1 = X_1 + X_2, \) where \( X_1 \) and \( X_2 \) have the joint pdf \( f_{X_1,X_2}(x_1, x_2) = 2e^{-(x_1+x_2)}, 0 < x_1 < x_2 < \infty, \) zero elsewhere.

Solution.

Since the support of \( Y_1 \) and \( Y_2 \) is \( 0 < y_1 - y_2 < \infty, 0 < y_2 < \infty \Rightarrow 0 < y_1/2 < y_2 < y_1 < \infty, \)

\[ f_Y(y_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(y_1 - y_2, y_2)dy_2 = \int_{y_1/2}^{y_1} 2e^{-y_1}dy_2 = y_1e^{-y_1}, \quad y_1 > 0, \]

which means \( Y \sim \text{Exp}(1). \)

2.2.8. Suppose \( X_1 \) and \( X_2 \) have the joint pdf

\[ f(x_1, x_2) = \begin{cases} 
    e^{-x_1}e^{-x_2} & x_1 > 0, x_2 > 0 \\
    0 & \text{elsewhere}
\end{cases} 
\]

For constants \( w_1 > 0 \) and \( w_2 > 0, \) let \( W = w_1X_1 + w_2X_2. \)

(a) Show that the pdf of \( W \) is

\[ f(x_1, x_2) = \begin{cases} 
    \frac{1}{w_1-w_2}(e^{-w/w_1} - e^{-w/w_2}) & w > 0 \\
    0 & \text{elsewhere}
\end{cases} 
\]

Solution.

Let \( Z = w_1X_1 - w_2X_2. \) This is one-to-one transformation so that we have

\[ x_1 = \frac{w + z}{2w_1}, \quad x_2 = \frac{w - z}{2w_2}. \]

Then the Jacobian is given by

\[ J = \left| \begin{array}{cc}
    \frac{\partial x_1}{\partial w} & \frac{\partial x_1}{\partial z} \\
    \frac{\partial x_2}{\partial w} & \frac{\partial x_2}{\partial z}
\end{array} \right| = \left| \begin{array}{cc}
    1/2w_1 & 1/2w_1 \\
    1/2w_2 & -1/2w_2
\end{array} \right| = -\frac{1}{2w_1w_2}. \]

Hence the joint pdf of \( W \) and \( Z \) is

\[ f_{W,Z}(w, z) = f \left( \frac{w + z}{2w_1}, \frac{w - z}{2w_2} \right) |J| = e^{-\frac{w+z}{2w_1}}e^{-\frac{w-z}{2w_2}} \frac{1}{2w_1w_2} = \frac{1}{2w_1w_2} e^{\frac{w_1-w_2}{2w_1w_2} w} e^{\frac{w_1-w_2}{2w_1w_2} z}. \]

The support is

\[ \frac{w + z}{2w_1} > 0, \quad \frac{w - z}{2w_2} > 0 \Rightarrow w > 0, \quad -w < z < w. \]
Hence the marginal pdf of $W$ is

$$f_W(w) = \frac{1}{2w_1 w_2} e^{-\frac{w_1 + w_2}{2w_1 w_2}} w \int_0^w e^{\frac{w_1 - w z}{2w_1 w_2}} \, dz$$

$$= \frac{1}{w_1 - w_2} e^{-\frac{w_1 + w_2}{2w_1 w_2}} w \left[ e^{\frac{w_1 - w z}{2w_1 w_2}} \right]_{w}^{-w}$$

$$= \frac{1}{w_1 - w_2} e^{-\frac{w_1 + w_2}{2w_1 w_2}} w \left( e^{\frac{w_1 - w}{2w_1 w_2}} - e^{-\frac{w_1 - w z}{2w_1 w_2}} \right)$$

$$= \frac{1}{w_1 - w_2} \left( e^{-w/w_1} - e^{-w/w_2} \right), \quad w > 0.$$

(b) Verify that $f_W(w) > 0$ for $w > 0$.

**Solution.**

If $w_1 > w_2$, then $w_1 - w_2 > 0$, $e^{-w/w_1} - e^{-w/w_2} > 0$ because $g(x) = e^{-a/x}$ is increasing for $a > 0$.

If $w_1 < w_2$, then $w_1 - w_2 < 0$, $e^{-w/w_1} - e^{-w/w_2} < 0$. Hence, $f_W(w) > 0$ for $w > 0$.

(c) Note that the pdf $f_W(w)$ has an indeterminate form when $w_1 = w_2$. Rewrite $f_W(w)$ using $h$ defined as $w_1 - w_2 = h$. Then use l’Hopital’s rule to show that when $w_1 = w_2$, the pdf is given by $f_W(w) = (w/w_1^2) \exp \{-w/w_1\}$ for $w > 0$ and zero elsewhere.

**Solution.**

When $w_1 = w_2$, or equivalently $h \to 0$,

$$\lim_{h \to 0} f_W(w) = \lim_{h \to 0} \frac{e^{-w/w_1} - e^{-w/(w_1-h)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{d}{dh} \left( e^{-w/w_1} - e^{-w/(w_1-h)} \right)}{dh/dh}$$

$$= \lim_{h \to 0} \frac{0 + \{w/(w_1 - h)^2\} e^{-w/(w_1-h)}}{1}$$

$$= w/w_1^2 e^{-w/w_1}.$$

2.3 Conditional Distributions and Expectations

2.3.5. Let $X_1$ and $X_2$ be two random variables such that the conditional distributions and means exist. Show that:

(a) $E(X_1 + X_2|X_2) = E(X_1|X_2) + X_2$.

**Solution.**

Consider $X_2 = x_2$ (a fixed number) first.

$$E(X_1 + X_2|X_2 = x_2) = E(X_1|X_2 = x_2) + x_2 \Rightarrow E(X_1 + X_2|X_2) = E(X_1|X_2) + X_2.$$

(b) $E(u(X_2)|X_2) = u(X_2)$.

**Solution.** $E(u(X_2)|X_2 = x_2) = E(u(x_2)) = u(x_2) \Rightarrow E(u(X_2)|X_2) = u(X_2)$.

2.3.6. Let the joint pdf of $X$ and $Y$ be given by

$$f(x, y) = \begin{cases} \frac{2}{(1+x+y)^2} & 0 < x < \infty, \ 0 < y < \infty \\ 0 & \text{elsewhere}. \end{cases}$$
(a) Compute the marginal pdf of $X$ and the conditional pdf of $Y$, given $X = x$.

Solution.

$$f(x) = \int_0^\infty \frac{2}{(1 + x + y)^3} dy = \left[ -\frac{1}{(1 + x + y)^2} \right]_0^\infty = \frac{1}{(1 + x)^2}, \quad 0 < x < \infty,$$

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{2(1 + x)^2}{(1 + x + y)^3}, \quad 0 < x < \infty, \quad 0 < y < \infty,$$

zero elsewhere.

(b) For a fixed $X = x$, compute $E(1 + x + Y|x)$ and use the result to compute $E(Y|x)$.

Solution.

$$E(1 + x + Y|x) = \int_0^\infty (1 + x + y) \frac{2(1 + x)^2}{(1 + x + y)^3} dy = \int_0^\infty \frac{2(1 + x)^2}{(1 + x + y)^2} dy = \left[ \frac{-2(1 + x)^2}{y} \right]_0^\infty = 2(1 + x).$$

Since $E(1 + x + Y|x) = 1 + x + E(X|Y), E(Y|x) = 1 + x.$

2.3.7. Suppose $X_1$ and $X_2$ are discrete random variables which have the joint pmf $p(x_1, x_2) = (3x_1 + x_2)/24, (x_1, x_2) = (1, 1), (1, 2), (2, 1), (2, 2)$, zero elsewhere. Find the conditional mean $E(X_2|x_1)$, when $x_1 = 1$.

Solution.

$$E(X_2|x_1 = 1) = \sum_{x_2 \in \{1, 2\}} x_2 p(1, x_2) = p(1, 1) + 2p(2, 1) = \frac{4}{24} + \frac{5}{24} = \frac{7}{12}.$$

2.3.8. Let $X$ and $Y$ have the joint pdf $f(x, y) = 2 \exp\{-x + y\}, 0 < x < y < \infty$, zero elsewhere. Find the conditional mean $E(Y|x)$ of $Y$, given $X = x$.

Solution.

$$f(x) = \int_x^\infty 2 \exp\{-x + y\} dy = 2e^{-2x} \Rightarrow f_{2|x}(y|x) = \frac{f(x,y)}{f(x)} = e^{x-y}, \quad 0 < x < y < \infty.$$

Hence,

$$E(Y|x) = \int_x^\infty ye^{x-y} dy = \int_0^\infty (x + t)e^{-t} dt = x + 1, \quad x > 0.$$

2.3.10. Let $X_1$ and $X_2$ have the joint pmf $p(x_1, x_2)$ described as follows:

<table>
<thead>
<tr>
<th>$(x_1, x_2)$</th>
<th>$(0, 0)$</th>
<th>$(0, 1)$</th>
<th>$(1, 0)$</th>
<th>$(1, 1)$</th>
<th>$(2, 0)$</th>
<th>$(2, 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x_1, x_2)$</td>
<td>$\frac{1}{18}$</td>
<td>$\frac{2}{18}$</td>
<td>$\frac{3}{18}$</td>
<td>$\frac{1}{18}$</td>
<td>$\frac{3}{18}$</td>
<td>$\frac{1}{18}$</td>
</tr>
</tbody>
</table>

and $p(x_1, x_2)$ is equal to zero elsewhere. Find the two marginal probability mass functions and the two conditional means.

*Hint:* Write the probabilities in a rectangular array.

Solution.

$$p(x_1) = \begin{cases} \frac{11}{18} & x_2 = 0, \\ \frac{2}{18} & x_2 = 1, \end{cases}, \quad p(x_2) = \begin{cases} \frac{4}{18} & x_1 = 0, \\ \frac{7}{18} & x_1 = 1, \\ \frac{7}{18} & x_1 = 2. \end{cases}$$

$$E(X_1|X_2 = x_2) = \begin{cases} \frac{16}{18} & x_2 = 0, \\ \frac{5}{18} & x_2 = 1, \end{cases}, \quad E(X_2|X_1 = x_1) = \begin{cases} \frac{3}{18} & x_1 = 0, \\ \frac{3}{18} & x_1 = 1, \\ \frac{1}{18} & x_1 = 2. \end{cases}$$
2.3.11. Let us choose at random a point from the interval \((0,1)\) and let the random variable \(X_1\) be equal to the number that corresponds to that point. Then choose a point at random from the interval \((0,x_1)\), where \(x_1\) is the experimental value of \(X_1\); and let the random variable \(X_2\) be equal to the number that corresponds to this point.

(a) Make assumptions about the marginal pdf \(f_1(x_1)\) and the conditional pdf \(f_{2|1}(x_2|x_1)\).

**Solution.**

Assume that \(X_1 \sim U(0,1)\) and \(X_2|X_1 = x_1 \sim U(0,x_2)\):

\[
f(x_1) = I(0 < x_1 < 1), \quad f(x_2|x_1) = \frac{1}{x_1} I(0 < x_2 < x_1).
\]

(b) Compute \(P(X_1 + X_2 \geq 1)\).

**Solution.**

By (a), \(f_{1,2}(x_1, x_2) = f(x_2|x_1)f(x_1) = 1/x_1, 0 < x_2 < x_1 < 1\). Hence,

\[
P(X_1 + X_2 \geq 1) = P(X_2 \geq 1 - X_1) = \int_{1/2}^1 \int_{1-x}^{x_1} \frac{1}{x_1} dx_2 dx_1 = \int_{1/2}^1 (2 - \frac{1}{x_1}) dx_1 = 1 - \log 2.
\]

(c) Find the conditional mean \(E(X_1|x_2)\)

**Solution.**

Find \(f(x_2)\) to get \(f(x_1|x_2)\).

\[
f(x_2) = \int_{x}^{1} \frac{1}{x_1} dx_1 = -\log x_2, \quad 0 < x_2 < 1 \Rightarrow f(x_1|x_2) = \frac{f(x_1,x_2)}{f(x_2)} = -\frac{1}{x_1 \log x_2}, \quad 0 < x_2 < x_1 < 1.
\]

Hence,

\[
E(X_1|X_2 = x_2) = \int_{x_2}^{1} -\frac{1}{\log x_2} dx_1 = \frac{1 - x_2}{\log(1/x_2)}, \quad 0 < x_2 < 1.
\]

2.3.12. Let \(f(x)\) and \(F(x)\) denote, respectively, the pdf and the cdf of the random variable \(X\). The conditional pdf of \(X\), given \(X > x_0\), \(x_0\) a fixed number, is defined by \(f(x|X > x_0) = f(x)/[1 - F(x_0)]\), \(x_0 < x\), zero elsewhere. This kind of conditional pdf finds application in a problem of time until death, given survival until time \(x_0\).

(a) Show that \(f(x|X > x_0)\) is a pdf.

**Solution.**

Since \(f(x) > 0\) and \(0 < F(x) < 1\), \(f(x|X > x_0) = f(x)/[1 - F(x_0)] > 0\). Also,

\[
\int_{x_0}^{\infty} f(x|X > x_0)dx = \int_{x_0}^{\infty} \frac{f(x)}{[1 - F(x_0)]} dx = \frac{1}{[1 - F(x_0)]}(F(x)|_{x_0}^{\infty} = 1 \quad \text{since} \quad F(\infty) = 1.
\]

(b) Let \(f(x) = e^{-x}, \quad 0 < x < \infty, \) and zero elsewhere. Compute \(P(X > 2|X > 1)\).

**Solution.**

Since \(F(x) = 1 - e^{-x}, \ x > 0, \ f(x|X > 1) = f(x)/[1 - F(1)] = e^{-x+1}\). Hence,

\[
P(X > 2|X > 1) = \int_{2}^{\infty} f(x|X > 1)dx = \int_{2}^{\infty} e^{-x+1} dx = [-e^{-x+1}]_{2}^{\infty} = e^{-1}.
\]
2.4 Independent Random Variables

2.4.1. Show that the random variables $X_1$ and $X_2$ with joint pdf

$$f(x_1, x_2) = \begin{cases} 12x_1x_2(1 - x_2) & 0 < x_1 < 1, 0 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

are independent.

Solution.

The support is rectangular (a product space). And $f(x_1, x_2)$ can be written as a product of a nonnegative function of $x_1$ and a nonnegative function of $x_2$: $f(x_1, x_2) = g(x_1)h(x_2)$, where $g(x_1) = 12x_1(0 < x_1 < 1)$ and $h(x_2) = x_2(1 - x_2)I(0 < x_2 < 1)$. Thus, $X_1$ and $X_2$ are independent.

Another solution is $f(x_1, x_2) = f(x_1)f(x_2)$, where $f(x_1) = 2x_1$ and $f(x_2) = 6x_2(1 - x_2)$ are marginal pdfs of $X_1$ and $X_2$.

2.4.2. If the random variables $X_1$ and $X_2$ have the joint pdf $f(x_1, x_2) = 2e^{-x_1-x_2}$, $0 < x_1 < x_2$, zero elsewhere, show that $X_1$ and $X_2$ are dependent.

Solution.

Although the joint pdf can be expressed by a product of two nonnegative functions of $x_1$ and $x_2$, respectively, $0 < x_1 < x_2 < \infty$ is not a product space, which implies that $X_1$ and $X_2$ are dependent.

2.4.3. Let $p(x_1, x_2) = \frac{1}{16}$, $x_1 = 1, 2, 3, 4$, and $x_2 = 1, 2, 3, 4$, zero elsewhere, be the joint pmf of $X_1$ and $X_2$. Show that $X_1$ and $X_2$ are independent.

Solution.

The marginal pdfs of $X_1$ and $X_2$ are $p(x_1) = p(x_2) = 1/4$. So $p(x_1, x_2) = p(x_1)p(x_2)$ and the space is rectangular, which gives us $X_1$ and $X_2$ are independent.

2.4.4. Find $P(0 < X_1 < \frac{1}{3}, 0 < X_2 < \frac{1}{3})$ if the random variables $X_1$ and $X_2$ have the joint pdf $f(x_1, x_2) = 4x_1(1 - x_2)$, $0 < x_1 < 1, 0 < x_2 < 1$, zero elsewhere.

Solution.

Since $f(x_1) = 2x_1$, $0 < x_1 < 1$ and $f(x_2) = 2(1 - x_2)$, $0 < x_2 < 1$, and $X_1$ and $X_2$ are independent,

$$P \left( 0 < X_1 < \frac{1}{3}, 0 < X_2 < \frac{1}{3} \right) = P(0 < X_1 < \frac{1}{3})P(0 < X_2 < \frac{1}{3}) = \left( \int_0^{1/3} 2x_1 \, dx_1 \right) \left( \int_0^{1/3} 2(1 - x_2) \, dx_2 \right) = \left( \frac{1}{9} \right) \left( \frac{5}{9} \right) = \frac{5}{81}.$$

2.4.5. Find the probability of the union of the events $a < X_1 < b$, $-\infty < X_2 < \infty$, and $-\infty < X_1 < \infty$, $c < X_2 < d$ if $X_1$ and $X_2$ are two independent variables with $P(a < X_1 < b) = \frac{2}{3}$ and $P(c < X_2 < d) = \frac{5}{8}$.

Solution.

$$P\left( \{ a < X_1 < b, \infty < X_2 < \infty \} \cup \{ -\infty < X_1 < \infty, \ c < X_2 < d \} \right)$$

$$= P(\{ a < X_1 < b \} \cup \{ c < X_2 < d \})$$

$$= P(a < X_1 < b) + P(c < X_2 < d) - P(\{ a < X_1 < b \} \cap \{ c < X_2 < d \})$$

$$= P(a < X_1 < b) + P(c < X_2 < d) - P(a < X_1 < b)P(c < X_2 < d)$$

$$= \frac{2}{3} + \frac{5}{8} - \left( \frac{5}{8} \right) = \frac{7}{8}.$$
2.4.8. Let $X$ and $Y$ have the joint pdf $f(x,y) = 3x$, $0 < y < x < 1$, zero elsewhere. Are $X$ and $Y$ independent? If not, find $E(X|y)$. 

Solution.

$X$ and $Y$ are not independent because the support $0 < y < x < 1$ is not rectangular (not a product space). So find $f(y)$ first: $f(y) = \int_0^1 3x \, dx = 3(1-y^2)/2$, $0 < y < 1$, zero elsewhere. Hence

$$E(X|y) = \int_{-\infty}^{\infty} x \frac{f(x,y)}{f(y)} \, dx = \int_y^1 \frac{2x^2}{(1-y^2)} \, dx = \frac{2(1-y^3)}{3(1-y^2)} = \frac{2(1+y+y^2)}{3(1+y)}, \ 0 < y < 1.$$

2.4.10. Let $X$ and $Y$ be random variables with the space consisting of the four points $(0,0)$, $(1,1)$, $(1,0)$, $(1,-1)$. Assign positive probabilities to these four points so that the correlation coefficient is equal to zero. Are $X$ and $Y$ independent?

Solution.

Assume the uniform distribution as shown below:

<table>
<thead>
<tr>
<th>$x_1$, $x_2$</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>$p_{X_1}(x_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1/4</td>
<td>0</td>
<td>1/4</td>
</tr>
<tr>
<td>1</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
<td>3/4</td>
</tr>
</tbody>
</table>

$p_{X_2}(x_2) = 1/4$ for $x_2 = 1$, zero elsewhere. Hence $E(X) = 3/4$, $E(Y) = 0$, $E(XY) = -1/4 + 1/4 = 0 \Rightarrow E(XY) - E(X)E(Y) = 0$. However, $P(X_1 = X_2 = 1) = 1/4 \neq 3/16 = p_{X_1}(1)p_{X_2}(1)$, meaning that $X$ and $Y$ are not independent.

2.4.11. Two line segments, each of length two units, are placed along the $x$-axis. The midpoint of the first is between $x = 0$ and $x = 14$ and that of the second is between $x = 6$ and $x = 20$. Assuming independence and uniform distributions for these midpoints, find the probability that the line segments overlap.

Solution.

Since $X_1 \sim U(0,14)$ and $X_2 \sim U(6,20)$, the joint pdf of $X_1$ and $X_2$ is $f(x_1,x_2) = 1/14^2$. The desired probability is

$$P(X_1 \geq X_2) = \int_6^{14} \int_6^{x_1} \frac{1}{14^2} \, dx_2 \, dx_1 = \left[ \frac{(x_1-6)^2}{2(14^2)} \right]_6^{14} = \frac{8}{49}.$$

2.4.12. Cast a fair die and let $X = 0$ if 1, 2, or 3 spots appear, let $X = 1$ if 4 or 5 spots appear, and let $X = 2$ if 6 spots appear. Do this two independent times, obtaining $X_1$ and $X_2$. Calculate $P(|X_1 - X_2| = 1)$.

Solution.

$|X_1 - X_2| = 1$ when $(X_1, X_2) = (0,1), (1,0), (1,2), (2,1)$ with probabilities of $1/6$, $1/6$, $1/18$, and $1/18$, respectively. Hence the desired probability is $2(1/6 + 1/18) = 4/9$.

2.4.13. For $X_1$ and $X_2$ in Example 2.4.6, show that the mgf of $Y = X_1 + X_2$ is $e^{2t}/(2-e^t)^2$, $t < \log 2$, and then compute the mean and variance of $Y$.

Solution.

Let $t = t_1 = t_2$ then

$$M_Y(t) = M_{X_1,X_2}(t,t) = \left( \frac{e^t}{2-e^t} \right)^2 = \frac{e^{2t}}{(2-e^t)^2}, \ t < \log 2.$$
Let $\psi(t) = \log M_Y(t) = 2t - 2\log(2 - e^t)$. Then

\[
E(Y) = \psi'(0) = 2 + \frac{2e^t}{2 - e^t}\bigg|_{t=0} = 4,
\]

\[
\text{Var}(Y) = \psi''(0) = \frac{4e^t}{(2 - e^t)^2}\bigg|_{t=0} = 4.
\]

### 2.5. The Correlation Coefficient

#### 2.5.1. Let the random variables $X$ and $Y$ have the joint pmf

(a) $p(x, y) = \frac{1}{3}$, $(x, y) = (0, 0), (1, 1), (2, 2)$, zero elsewhere.

(b) $p(x, y) = \frac{1}{3}$, $(x, y) = (0, 2), (1, 1), (2, 0)$, zero elsewhere.

(c) $p(x, y) = \frac{1}{3}$, $(x, y) = (0, 0), (1, 1), (2, 0)$, zero elsewhere.

In each case compute the correlation coefficient of $X$ and $Y$.

**Solution.**

For (a) and (b), the scatter plots clearly show that $\rho = 1$ and $\rho = -1$, respectively.

For (c), since $E(X) = 1, E(Y) = \frac{1}{3}$, and $E(XY) = \frac{1}{3}$, Cov($X, Y$) = $E(XY) - E(X)E(Y) = 0$. Thus, $\rho = 0$.

#### 2.5.3. Let $f(x, y) = 2, 0 < x < y, 0 < y < 1$, zero elsewhere, be the joint pdf of $X$ and $Y$. Show that the conditional means are, respectively, $(1 + x)/2, 0 < x < 1$, and $y/2, 0 < y < 1$. Show that the correlation coefficient of $X$ and $Y$ is $\rho = \frac{1}{2}$.

**Solution.**

Find the marginal pdfs of $X$ and $Y$ first.

\[
f(x) = \int_x^1 2dy = 2(1 - x), \; 0 < x < 1, \quad f(y) = \int_0^y 2dx = 2y, \; 0 < y < 1.
\]

Hence,

\[
E(Y|X = x) = \int_{-\infty}^\infty yf(y|x)dy = \int_{-\infty}^\infty \frac{yf(x,y)}{f(x)}dy = \int_x^1 \frac{y}{1 - x}dy = \frac{1 + x}{2}, \; 0 < x < 1,
\]

\[
E(X|Y = y) = \int_{-\infty}^\infty xf(x|y)d\!x = \int_{-\infty}^\infty \frac{xf(x,y)}{f(y)}dy = \int_0^y \frac{x}{y}dy = \frac{y}{2}, \; 0 < y < 1.
\]

#### 2.5.4. Show that the variance of the conditional distribution of $Y$, given $X = x$, in Exercise 2.5.3, is $(1 - x)^2/12, 0 < x < 1$, and that the variance of the conditional distribution of $X$, given $Y = y$, is $y^2/12, 0 < y < 1$.

**Solution.**

\[
E(Y^2|X = x) = \int_x^1 \frac{y^2}{1 - x}dy = \frac{1 + x + x^2}{3}, \; 0 < x < 1,
\]

\[
E(X^2|Y = y) = \int_0^y \frac{x^2}{y}dy = \frac{y^2}{3}, \; 0 < y < 1.
\]

Hence,

\[
\text{Var}(Y|X = x) = E(Y^2|X = x) - [E(Y|X = x)]^2 = \frac{1 + x + x^2}{3} - \left(\frac{1 + x}{2}\right)^2 = \frac{(1 - x)^2}{12}, \; 0 < x < 1,
\]

\[
\text{Var}(X|Y = y) = E(X^2|Y = y) - [E(X|Y = y)]^2 = \frac{y^2}{3} - \frac{y^2}{4} = \frac{y^2}{12}, \; 0 < y < 1.
\]
2.5.5. Verify the results of equations (2.5.11) of this section.

Solution. See Exercise 2.5.8 because using $\psi(t_1, t_2)$ is easier to compute them.

2.5.6. Let $X$ and $Y$ have the joint pdf $f(x, y) = 1, -x < y < x, 0 < x < 1$, zero elsewhere. Show that, on the set of positive probability density, the graph of $E(Y | x)$ is a straight line, whereas that of $E(X | y)$ is not a straight line.

Solution.

Find the marginal pdfs of $X$ and $Y$ first.

$$f(x) = \int_{-x}^{x} dy = 2x, \ 0 < x < 1; \quad f(y) = \begin{cases} \int_{y}^{1} dx = 1 - y & 0 < y < 1 \\ \int_{0}^{1} dx = 1 & -1 < y \leq 0 \end{cases}.$$ 

Hence,

$$E(Y | x) = \int_{-\infty}^{\infty} y f(y | x) dy = \int_{-\infty}^{\infty} y f(x, y) f(x) dy = \int_{-\infty}^{\infty} \frac{y}{2x} dy = 0, \quad 0 < x < 1,$$

$$E(X | y) = \int_{-\infty}^{\infty} x f(x | y) dx = \int_{-\infty}^{\infty} x f(x, y) f(y) dy = \begin{cases} \int_{0}^{1} \frac{x}{1 - y} dy = \frac{1 + y}{2} & 0 < y < 1 \\ \int_{0}^{1} x dy = \frac{1}{2} & -1 < y \leq 0 \end{cases}$$

which means that the graph of $E(Y | x)$ is a straight line, whereas that of $E(X | y)$ is not a straight line.

2.5.8. Let $\psi(t_1, t_2) = \log M(t_1, t_2)$, where $M(t_1, t_2)$ is the mgf of $X$ and $Y$. Show that

$$\frac{\partial \psi(0, 0)}{\partial t_i} = \frac{\partial^2 \psi(0, 0)}{\partial t_i^2}, \quad i = 1, 2,$$

and

$$\frac{\partial^2 \psi(0, 0)}{\partial t_1 \partial t_2}$$

yield the means, the variances, and the covariance of the two random variables. Use this result to find the means, the variances, and the covariance of $X$ and $Y$ of Example 2.5.6.

Solution.

Note that $M(0, 0) = E(1) = 1$. When $i = 1,$

$$\frac{\partial \psi(0, 0)}{\partial t_1} = \frac{\partial M(0, 0) / \partial t_1}{M(0, 0)} = \int_{-\infty}^{\infty} x f(x, y) dx = \int_{-\infty}^{\infty} x f(x) dx = E(X),$$

$$\frac{\partial^2 \psi(0, 0)}{\partial t_1^2} = \frac{M(0, 0) \partial^2 M(0, 0) / \partial t_1^2 - [\partial M(0, 0) / \partial t_1]^2}{M(0, 0)^2} = E(X^2) - [E(X)]^2 = \text{Var}(X).$$

Same for $i = 2.$ And

$$\frac{\partial^2 \psi(0, 0)}{\partial t_1 \partial t_2} = \frac{\partial^2 M(0, 0) / \partial t_1 \partial t_2}{M(0, 0)}$$

$$= \frac{[\partial^2 M(0, 0) / \partial t_1 \partial t_2] M(0, 0) - [\partial M(0, 0) / \partial t_1][\partial M(0, 0) / \partial t_2]}{M(0, 0)^2}$$

$$= E(XY) - E(X)E(Y) = \text{Cov}(X, Y).$$

Hence, for Example 2.5.6,

$$\psi(t_1, t_2) = \log M(t_1, t_2) = - \log(1 - t_1 - t_2) - \log(1 - t_2),$$

$$\frac{\partial \psi(t_1, t_2)}{\partial t_1} = \frac{1}{1 - t_1 - t_2}, \quad \frac{\partial \psi(t_1, t_2)}{\partial t_2} = \frac{1}{1 - t_1 - t_2} + \frac{1}{1 - t_2},$$

$$\frac{\partial^2 \psi(t_1, t_2)}{\partial t_1^2} = \frac{1}{(1 - t_1 - t_2)^2}, \quad \frac{\partial^2 \psi(t_1, t_2)}{\partial t_2^2} = \frac{1}{(1 - t_1 - t_2)^2} + \frac{1}{(1 - t_2)^2}$$

$$\frac{\partial^2 \psi(t_1, t_2)}{\partial t_1 \partial t_2} = \frac{1}{(1 - t_1 - t_2)^2}.$$
Therefore,

\[ \mu_1 = E(X) = \frac{\partial \psi(0,0)}{\partial t_1} = 1, \quad \mu_2 = E(Y) = \frac{\partial \psi(0,0)}{\partial t_2} = 2 \]

\[ \sigma_1^2 = \text{Var}(X) = \frac{\partial^2 \psi(0,0)}{\partial t_1^2} = 1, \quad \sigma_2^2 = \text{Var}(Y) = \frac{\partial^2 \psi(0,0)}{\partial t_2^2} = 2 \]

\[ E[(X - \mu_1)(Y - \mu_2)] = \text{Cov}(X,Y) = \frac{\partial^2 \psi(0,0)}{\partial t_1 \partial t_2} = 1. \]

2.5.9. Let \( X \) and \( Y \) have the joint pmf \( p(x,y) = \frac{1}{7}, (0,0), (1,0), (0,1), (1,1), (2,1), (1,2), (2,2), \) zero elsewhere. Find the correlation coefficient \( \rho \).

Solution.

\[ E(X) = E(Y) = \frac{1+1+2+1+2}{7} = 1, \quad E(X^2) = E(Y^2) = \frac{1+1+4+1+4}{7} = \frac{11}{7} \]

\[ \Rightarrow \sigma_X^2 = \sigma_Y^2 = \frac{11}{7} - 1 = \frac{4}{7}, \quad E(XY) = \frac{1+2+2+4}{7} = \frac{9}{7}. \]

Hence,

\[ \rho = \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y} = \frac{2/7}{4/7} = \frac{1}{2}. \]

2.5.11. Let \( \sigma_1^2 = \sigma_2^2 = \sigma^2 \) be the common variance of \( X_1 \) and \( X_2 \) and let \( \rho \) be the correlation coefficient of \( X_1 \) and \( X_2 \). Show for \( k > 0 \) that

\[ P[|(X_1 - \mu_1) + (X_2 - \mu_2)| \geq k\sigma] \leq \frac{2(1+\rho)}{k^2}. \]

Solution.

\[ P[|(X_1 - \mu_1) + (X_2 - \mu_2)| \geq k\sigma] = P[|(X_1 - \mu_1) + (X_2 - \mu_2)|^2 \geq k^2\sigma^2] \]

\[ = P[(X_1 - \mu_1)^2 + (X_2 - \mu_2)^2 + 2(X_1 - \mu_1)(X_2 - \mu_2) \geq k^2\sigma^2] \]

\[ \leq P[(X_1 - \mu_1)^2 \geq k^2\sigma^2] + P[(X_2 - \mu_2)^2 \geq k^2\sigma^2] \]

\[ + P[2(X_1 - \mu_1)(X_2 - \mu_2) \geq k^2\sigma^2] \]

\[ = P|(X_1 - \mu_1)| \geq k\sigma + P|(X_2 - \mu_2)| \geq k\sigma \]

\[ + P[2(X_1 - \mu_1)(X_2 - \mu_2) \geq k^2\sigma^2] \]

\[ \leq \frac{1}{k^2} + \frac{1}{k^2} + \frac{2E(X_1 - \mu_1)(X_2 - \mu_2)}{k^2\sigma^2} \]

\[ = \frac{2(1+\rho)}{k^2 \sigma^2} \text{ since } E(X_1 - \mu_1)(X_2 - \mu_2) = \rho. \]

2.6. Extension to Several Random Variables

2.6.1. Let \( X, Y, Z \) have joint pdf \( f(x,y,z) = \frac{2(x+y+z)}{3}, 0 < x < 1, 0 < y < 1, 0 < z < 1, \) zero elsewhere.

(a) Find the marginal probability density functions of \( X, Y, \) and \( Z \).

Solution.

\[ f_X(x) = \int_0^1 \int_0^1 \frac{2(x+y+z)}{3} \, dz \, dy = \cdots = \frac{2(x+1)}{3}. \]

Similarly,

\[ f_Y(y) = \frac{2(y+1)}{3}, \quad f_Z(z) = \frac{2(z+1)}{3}. \]
(b) Compute \(P(0 < X < \frac{1}{2}, 0 < Y < \frac{1}{2}, 0 < Z < \frac{1}{2})\) and \(P(0 < X < \frac{1}{2}) = P(0 < Y < \frac{1}{2}) = P(0 < Z < \frac{1}{2})\).

**Solution.** Skipped. We can solve part (c) without computing them.

(c) Are \(X, Y,\) and \(Z\) independent?

**Solution.** No; \(f(x, y, x) \neq f(x)f(y)f(z)\) although the support is a product space.

(d) Compute \(E(X^2YZ + 3XY^4Z^2)\).

**Solution.** Skipped.

(e) Determine the cdf of \(X, Y,\) and \(Z\).

**Solution.**

\[
F_X(x) = \begin{cases} 
0 & x \leq 0 \\
\int_0^x \frac{2(t+1)}{3} \, dt = \frac{(x+1)^2 - 1}{3} = \frac{x^2 + 2x}{3} & 0 < x < 1 \\
1 & x \geq 1 
\end{cases}
\]

Similarly,

\[
F_Y(y) = \begin{cases} 
0 & y \leq 0 \\
\frac{y^2 + 2y}{3} & 0 < y < 1 \\
1 & y \geq 1 
\end{cases},
\]

\[
F_Z(z) = \begin{cases} 
0 & z \leq 0 \\
\frac{z^2 + 2z}{3} & 0 < z < 1 \\
1 & z \geq 1 
\end{cases}
\]

(f) Find the conditional distribution of \(X\) and \(Y,\) given \(Z = z,\) and evaluate \(E(X + Y|z)\).

**Solution.**

\[
f(x, y|z) = \frac{f(x, y, z)}{f(z)} = \frac{x + y + z}{z + 1}, \quad 0 < x < 1, \quad 0 < y < 1.
\]

Hence,

\[
E(X + Y|z) = \int_0^1 \int_0^1 (x + y) \frac{x + y + z}{z + 1} \, dy \, dx
\]

\[
= \int_0^1 \frac{x + y + z}{z + 1} \left[ \int_0^{z+1} \frac{(x+y)^3 + z(x+y)^2}{3} \, dx \right] \, dy
\]

\[
= \frac{1}{z + 1} \left[ \int_0^{z+1} \left( \frac{(x+1)^3 + z(x+1)^2}{3} - \frac{x^2}{2} \right) \, dx \right]
\]

\[
= \frac{1}{z + 1} \left[ \frac{(x+1)^4}{12} + \frac{z(x+1)^3}{6} - \frac{x^4}{12} - \frac{zx^3}{6} \right]_0^{z+1}
\]

\[
= \frac{z + 7/6}{z + 1}, \quad 0 < z < 1.
\]

(g) Determine the conditional distribution of \(X,\) given \(Y = y\) and \(Z = z,\) and compute \(E(X|y, z)\).

**Solution.**

\[
f(y, z) = \int_0^1 \frac{2(x + y + z)}{3} \, dx = \frac{2y + 2z + 1}{3}
\]

\[
f(x|y, z) = \frac{f(x, y, z)}{f(y, z)} = \frac{2(x + y + z)}{2y + 2z + 1}.
\]
Hence,

\[ E(X|y,z) = \int_0^1 \frac{2(x+y+z)}{2y+2z+1} \, dx = \int_0^1 \frac{2x^2 + 2xy + z}{2y + 2z + 1} = \frac{3y + 3z + 2}{3(2y + 2z + 1)}, \quad 0 < y, z < 1. \]

2.6.2. Let \( f(x_1, x_2, x_3) = \exp[-(x_1 + x_2 + x_3)], 0 < x_1 < \infty, 0 < x_2 < \infty, 0 < x_3 < \infty, \) zero elsewhere, be the joint pdf of \( X_1, X_2, X_3. \)

(a) Compute \( P(X_1 < X_2 < X_3) \) and \( P(X_1 = X_2 < X_3). \)

Solution.

\[
P(X_1 < X_2 < X_3) = \int_0^\infty \int_0^{x_3} \int_0^{x_2} e^{-x_1-x_2-x_3} \, dx_1 \, dx_2 \, dx_3
\]

\[
= \int_0^\infty \int_0^{x_3} \left[ e^{-x_2-x_3} - e^{-2x_2-x_3} \right] \, dx_2 \, dx_3
\]

\[
= \int_0^\infty \left[ (e^{-x_3} - e^{-2x_3}) - (e^{-x_3}/2 - e^{-3x_3}/2) \right] \, dx_3
\]

\[
= (1 - 1/2) - (1/2 - 1/6) = \frac{1}{6},
\]

\[
P(X_1 = X_2 < X_3) = \int_0^\infty \int_0^{x_2} \int_0^{x_2} e^{-x_1-x_2-x_3} \, dx_1 \, dx_2 \, dx_3 = 0.
\]

(b) Determine the joint mgf of \( X_1, X_2, \) and \( X_3. \) Are these random variables independent?

Solution.

\[
M(t_1, t_2, t_3) = \int_0^\infty \int_0^\infty \int_0^\infty e^{-(1-t_1)x_1} e^{-(1-t_2)x_2} e^{-(1-t_3)x_3} \, dx_1 \, dx_2 \, dx_3
\]

\[
= \int_0^\infty e^{-(1-t_1)x_1} \, dx_1 \int_0^\infty e^{-(1-t_2)x_2} \, dx_2 \int_0^\infty e^{-(1-t_3)x_3} \, dx_3
\]

\[
= \frac{1}{(1-t_1)(1-t_2)(1-t_3)}, \quad t_1 < 1, t_2 < 1, t_3 < 1
\]

\[
= M_{X_1}(t_1)M_{X_2}(t_2)M_{X_3}(t_3),
\]

which clearly shows that these three random variables are independent.

2.6.7. Prove Corollary 2.6.1: Suppose \( X_1, X_2, \ldots, X_n \) are iid random variables with the common mgf \( M(t), \)

for \(-h < t < h, \) where \( h > 0. \) Let \( T = \sum_{i=1}^n X_i. \) Then \( T \) has the mgf given by

\[
M_T(t) = [M(t)]^n, \quad -h < t < h.
\]

Solution.

\[
M_T(t) = E\left[ e^{\sum_{i=1}^n X_i t} \right] = \prod_{i=1}^n E(e^{X_i t}) \quad (X_i's \ are \ independent)
\]

\[
= [E(e^{X_1 t})]^n \quad (X_i's \ are \ identical)
\]

\[
= [M_X(t)]^n.
\]

2.6.9. Let \( X_1, X_2, X_3 \) be iid with common pdf \( f(x) = \exp(-x), 0 < x < \infty, \) zero elsewhere. Evaluate:

(a) \( P(X_1 < X_2|X_1 < 2X_2). \)

Solution.

\[
P(X_1 < X_2|X_1 < 2X_2) = \frac{P(X_1 < X_2, X_1 < 2X_2)}{P(X_1 < 2X_2)} = \frac{P(X_1 < X_2)}{P(X_1 < 2X_2)}.
\]
For the numerator,
\[ P(X_1 < X_2) = \int_0^\infty \int_1^\infty e^{-x_1-x_2} dx_2 dx_1 = \int_0^\infty e^{-2x_1} dx_1 = \frac{1}{2}. \]

For the denominator,
\[ P(X_1 < 2X_2) = \int_0^\infty \int_{x_1/2}^\infty e^{-x_1-x_2} dx_2 dx_1 = \int_0^\infty e^{-3x_1/2} dx_1 = \frac{2}{3}. \]

Thus, \( P(X_1 < X_2|X_1 < 2X_2) = \frac{1/2}{2/3} = \frac{3}{4}. \)

(b) \( P(X_1 < X_2 < X_3|X_3 < 1). \)

Solution.
\[ P(X_1 < X_2 < X_3|X_3 < 1) = \frac{P(X_1 < X_2 < X_3 < 1)}{P(X_3 < 1)}. \]

For the numerator,
\[
\begin{align*}
P(X_1 < X_2 < X_3 < 1) &= \int_0^1 \int_0^{x_3} \int_0^{x_2} e^{-x_1-x_2-x_3} dx_1 dx_2 dx_3 \\
&= \int_0^1 \int_0^{x_3} [e^{-x_2-x_3} - e^{-2x_2-x_3}] dx_2 dx_3 \\
&= \int_0^1 [(e^{-x_3} - e^{-2x_3}) - (e^{-x_3}/2 - e^{-3x_3}/2)] dx_3 \\
&= \int_0^1 [e^{-x_3}/2 - e^{-2x_3} + e^{-3x_3}/2] dx_3 \\
&= \frac{1-e^{-1}}{2} - \frac{1-e^{-2}}{2} + \frac{1-e^{-3}}{6} \\
&= \frac{1-3e^{-1}+3e^{-2}-e^{-3}}{6}
\end{align*}
\]

For the denominator,
\[ P(X_3 < 1) = \int_0^1 e^{-x_3} dx_3 = 1 - e^{-1}. \]

Hence
\[ P(X_1 < X_2 < X_3|X_3 < 1) = \frac{P(X_1 < X_2 < X_3 < 1)}{P(X_3 < 1)} = \frac{1-3e^{-1}+3e^{-2}-e^{-3}}{6(1-e^{-1})} \approx 0.0666. \]

2.7. Transformations for Several Random Variables

Skipped because of a just extension from two random variables.

2.8. Linear Combinations of Random Variables

2.8.3. Let \( X_1 \) and \( X_2 \) be two independent random variables so that the variances of \( X_1 \) and \( X_2 \) are \( \sigma_1^2 = k \) and \( \sigma_2^2 = 2 \), respectively. Given that the variance of \( Y = 3X_2 - X_1 \) is 25, find \( k \).

Solution.
\[
\text{Var}(Y) = 3^2 \text{Var}(X_2) + \text{Var}(X_1) \quad X_1, X_2 \text{ are independent}
\]
\[ = 9\sigma_2^2 + \sigma_1^2 = 18 + k. \]
Hence, Var(Y) = 25 ⇒ k = 7.

2.8.6. Determine the mean and variance of the sample mean \( X = \frac{1}{5} \sum_{i=1}^{5} X_i \), where \( X_1, \ldots, X_5 \) is a random sample from a distribution having pdf \( f(x) = 4x^3 \), \( 0 < x < 1 \), zero elsewhere.

Solution.

\[
E(X) = \int_0^1 x(4x^3)dx = \frac{4}{5}, \quad E(X^2) = \int_0^1 x^2(4x^3)dx = \frac{2}{3} \Rightarrow Var(X) = \frac{2}{75}.
\]

Hence,

\[
E(\bar{X}) = E(X) = \frac{4}{5} = 0.8, \quad Var(\bar{X}) = \frac{Var(X)}{5} = \frac{2}{375} \approx 0.00533.
\]

2.8.7. Let \( X \) and \( Y \) be random variables with \( \mu_1 = 1, \mu_2 = 4, \sigma_1^2 = 4, \sigma_2^2 = 6, \rho = \frac{1}{2} \). Find the mean and variance of the random variable \( Z = 3X - 2Y \).

Solution.

\[
E(Z) = 3E(X) - 2E(Y) = 3\mu_1 - 2\mu_2 = -5
\]

\[
Var(Z) = 3^2Var(X) + 2^2Var(Y) - 12Cov(X,Y)
= 9\sigma_1^2 + 4\sigma_2^2 - 12\rho\sigma_1\sigma_2
= 60 - 12\sqrt{6} \approx 30.6.
\]

2.8.8. Let \( X \) and \( Y \) be independent random variables with means \( \mu_1, \mu_2 \) and variances \( \sigma_1^2, \sigma_2^2 \). Determine the correlation coefficient of \( X \) and \( Z = X - Y \) in terms of \( \mu_1, \mu_2, \sigma_1^2, \sigma_2^2 \).

Solution.

Since \( X \) and \( Y \) are independent,

\[
Var(Z) = Var(X) + Var(Y) = \sigma_1^2 + \sigma_2^2,
\]

\[
Cov(X, Z) = Cov(X, X - Y) = Var(X) - Cov(X, Y) = \sigma_1^2.
\]

Hence, the correlation coefficient is

\[
\rho = \frac{Cov(X, Z)}{\sqrt{Var(X)Var(Z)}} = \frac{\sigma_1^2}{\sqrt{\sigma_1^2(\sigma_1^2 + \sigma_2^2)}} = \frac{\sigma_1}{\sqrt{\sigma_1^2 + \sigma_2^2}}.
\]

2.8.10. Determine the correlation coefficient of the random variables \( X \) and \( Y \) if \( var(X) = 4, var(Y) = 2 \), and \( var(X + 2Y) = 15 \).

Solution.

\[
15 = Var(X + 2Y) = Var(X) + 4Var(Y) + 4Cov(X,Y) = 4 + 4(2) + 4\rho\sqrt{4\sqrt{2}} = 12 + 8\sqrt{2}\rho.
\]

Hence, \( \rho = 3/(8\sqrt{2}) \approx 0.265 \).

2.8.11. Let \( X \) and \( Y \) be random variables with means \( \mu_1, \mu_2 \); variances \( \sigma_1^2, \sigma_2^2 \); and correlation coefficient \( \rho \). Show that the correlation coefficient of \( W = aX + b, a > 0 \), and \( Z = cY + d, c > 0 \), is \( \rho \).

Solution.

\[
Var(W) = a^2Var(X) = a^2\sigma_1^2, \quad Var(Z) = c^2Var(Y) = c^2\sigma_2^2, \quad Cov(W, Z) = acCov(X, Y) = ac\rho\sigma_1\sigma_2.
\]

Hence, \( \text{Corr}(W, Z) = Cov(W, Z)/(\sqrt{Var(W)Var(Z)}) = \rho \) because \( a > 0 \) and \( c > 0 \).
2.8.13. Let $X_1$ and $X_2$ be independent random variables with nonzero variances. Find the correlation coefficient of $Y = X_1X_2$ and $X_1$ in terms of the means and variances of $X_1$ and $X_2$.

Solution.

Let $\mu_1$, $\mu_2$ and $\sigma_1^2$, $\sigma_2^2$ denote the means and the variances of $X_1$ and $X_2$, respectively. Since the two r.v.s. are independent,

$$\text{Var}(Y) = \text{Var}(X_1X_2)$$
$$= E(X_1^2X_2^2) - E(X_1X_2)^2$$
$$= E(X_1^2)E(X_2^2) - E(X_1)^2E(X_2)^2$$
$$= (\mu_1^2 + \sigma_1^2)(\mu_2^2 + \sigma_2^2) - \mu_1^2\mu_2^2$$
$$= \mu_1^2\sigma_2^2 + \sigma_1^2\mu_2^2 + \sigma_1\sigma_2\mu_1\mu_2,$$

$$\text{Cov}(Y, X_1) = \text{Cov}(X_1X_2, X_1)$$
$$= E(X_1^2X_2) - E(X_1X_2)E(X_1)$$
$$= E(X_1^2)E(X_2) - E(X_1)^2E(X_2)$$
$$= (\mu_1^2 + \sigma_1^2)\mu_2 - \mu_1^2\mu_2$$
$$= \sigma_1^2\mu_2.$$

Hence,

$$\rho = \frac{\text{Cov}(Y, X_1)}{\sqrt{\text{Var}(Y)\text{Var}(X_1)}} = \frac{\sigma_1^2\mu_2}{\sqrt{\mu_1^2\sigma_2^2 + \sigma_1^2\mu_2^2 + \sigma_1\sigma_2\mu_1\mu_2}} = \frac{\sigma_1\mu_2}{\sqrt{\mu_1^2\sigma_2^2 + \sigma_1^2\mu_2^2 + \sigma_1\sigma_2\mu_1\mu_2}}.$$

2.8.15. Let $X_1$, $X_2$, and $X_3$ be random variables with equal variances but with correlation coefficients $\rho_{12} = 0.3$, $\rho_{13} = 0.5$, and $\rho_{23} = 0.2$. Find the correlation coefficient of the linear functions $Y = X_1 + X_2$ and $Z = X_2 + X_3$.

Solution.

Let $\sigma^2$ denote the variance of $X_1$, $X_2$, and $X_3$. Then

$$\text{Var}(Y) = \text{Var}(X_1) + \text{Var}(X_2) + 2\text{Cov}(X_1, X_2) = 2\sigma^2(1 + \rho_{12}) = 2.6\sigma^2,$$

$$\text{Var}(Z) = \text{Var}(X_2) + \text{Var}(X_3) + 2\text{Cov}(X_2, X_3) = 2\sigma^2(1 + \rho_{23}) = 2.4\sigma^2,$$

$$\text{Cov}(Y, Z) = \text{Cov}(X_1 + X_2, X_2 + X_3) = \sigma^2(\rho_{12} + \rho_{13} + 1 + \rho_{23}) = 2\sigma^2.$$

Therefore, the correlation coefficient, $\rho$, is

$$\rho = \frac{\text{Cov}(Y, Z)}{\sqrt{\text{Var}(Y)\text{Var}(Z)}} = \frac{2\sigma^2}{\sqrt{2.6(2.4)\sigma^2}} \approx 0.801.$$

2.8.17. Let $X$ and $Y$ have the parameters $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$, and $\rho$. Show that the correlation coefficient of $X$ and $|Y - \rho(\sigma_2/\sigma_1)X|$ is zero.

Solution.

$$\text{Cov}(X, Y - \rho(\sigma_2/\sigma_1)X) = \text{Cov}(X, Y) - \rho(\sigma_2/\sigma_1)\text{Var}(X) = \rho\sigma_1\sigma_2 - \rho(\sigma_2/\sigma_1)\sigma_1^2 = 0.$$