Exercises in Introduction to Mathematical Statistics (Ch. 3)

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Note

- Not all solutions are provided: exercises that are too simple or not very important to me are skipped.
- Texts in red are just attentions to me. Please ignore them.

3 Some Special Distributions

3.1 The Binomial and Related Distributions

3.1.1. If the mgf of a random variable X is $(\frac{1}{3} + \frac{2}{3}e^t)^5$, find $P(X = 2 \text{ or } 3)$. Verify using the R function `dbinom`.

Solution.

Let $X \sim B(n, p)$. Then the mgf of X is given by

$$M_X(t) = \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x} = [(1-p) + pe^t]^n \quad \text{since } (a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x},$$

which gives $n = 5$ and $p = 2/3$ in this case. Hence,

$$P(X = 2 \text{ or } 3) = P(X = 2) + P(X = 3) = \binom{5}{2} \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^3 + \binom{5}{3} \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^2 = \frac{40}{81}.$$

3.1.4. Let the independent random variables $X_1, X_2, ..., X_{40}$ be iid with the common pdf $f(x) = 3x^2$, $0 < x < 1$, zero elsewhere. Find the probability that at least 35 of the X_i's exceed $\frac{1}{2}$.

Solution.

Since $F_X(x) = x^4, 0 < x < 1$, $P(X > 1/2) = 1 - F_X(1/2) = 7/8$. Hence, the desired probability is

$$\sum_{x=35}^{40} \binom{40}{x} \left(\frac{7}{8}\right)^x \left(\frac{1}{8}\right)^{n-x} = \texttt{1 - dbinom(34, 40, 7/8) = 0.6162}.$$

3.1.6. Let Y be the number of successes throughout n independent repetitions of a random experiment with probability of success $p = \frac{1}{4}$. Determine the smallest value of n so that $P(1 \le Y) \ge 0.70$.

Solution.

$$P(1 < Y) = 1 - P(Y = 0) = 1 - \left(\frac{3}{4}\right)^n \ge 0.70, \Rightarrow \left(\frac{3}{4}\right)^n \le 0.3.$$

Hence, $n = 5$ because $(3/4)^4 = 0.316 > 0.3 > (3/4)^5 = 0.237$.

3.1.7. Let the independent random variables X_1 and X_2 have binomial distribution with parameters $n_1 = 3$, $p = \frac{2}{3}$ and $n_2 = 4$, $p = \frac{1}{2}$, respectively. Compute $P(X_1 = X_2)$.

Solution.

Note that X_1 and X_2 are independent, then

$$P(X_1 = X_2) = \sum_{k=0}^{3} P(X_1 = X_2 = k) = \sum_{k=0}^{3} P(X_1 = k)P(X_2 = k) = \cdots = \frac{43}{144}.$$

3.1.11. Toss two nickels and three dimes at random. Make appropriate assumptions and compute the probability that there are more heads showing on the nickels than on the dimes.

Solution.

Let X_1 and X_2 denote the number of heads showing on the nickels and dimes, respectively. Assume that $X_1 \sim B(2, \frac{1}{2})$ and $X_2 \sim B(3, \frac{1}{2})$. Then

$$\begin{aligned} P(X_1 > X_2) &= P(X_1 = 1 \text{ or } 2, X_2 = 0) + P(X_1 = 2, X_2 = 1) \\ &= \left(\frac{1}{2} + \frac{1}{4}\right)\left(\frac{1}{8}\right) + \left(\frac{1}{4}\right)\left(\frac{3}{8}\right) = \frac{3}{16}. \end{aligned}$$

3.1.13. Let X be $b(2, p)$ and let Y be $b(4, p)$. If $P(X \geq 1) = \frac{5}{9}$, find $P(Y \geq 1)$.

Solution.

$$\frac{5}{9} = P(X \geq 1) = 1 - P(X = 0) = 1 - (1-p)^2 \;\Rightarrow\; p = \frac{1}{3}.$$

Thus,

$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - \left(\frac{2}{3}\right)^4 = \frac{65}{81}.$$

3.1.14. Let X have a binomial distribution with parameters n and $p = \frac{1}{3}$. Determine the smallest integer n can be such that $P(X \geq 1) \geq 0.85$.

Solution.

$$0.85 \leq P(X \geq 1) = 1 - P(X = 0) = 1 - (2/3)^n \;\Rightarrow\; (2/3)^n \leq 0.15,$$

which gives $n = 5$ because $(2/3)^4 = 0.20 > 0.15 > (2/3)^5 = 0.13$.

3.1.15. Let X have the pmf $p(x) = (\frac{1}{3})(\frac{2}{3})^x$, $x = 0, 1, 2, 3, ...$, zero elsewhere. Find the conditional pmf of X given that $X \geq 3$.

Solution.

$$P(X = x | X \geq 3) = \frac{P(X = x)}{P(X \geq 3)} = \frac{p(x)}{1 - p(0) - p(1) - p(2)} = \frac{(\frac{1}{3})(\frac{2}{3})^x}{(\frac{2}{3})^3} = \frac{1}{3}\left(\frac{2}{3}\right)^{x-3}, \quad x = 3, 4, 5, \ldots.$$

3.1.17. Show that the moment generating function of the negative binomial distribution is $M(t) = p^r[1 - (1-p)e^t]^{-r}$. Find the mean and the variance of this distribution.

Solution.

Let $X \sim \text{Geometric}(p)$ and $Y = \sum_{i=1}^{r} X_i$. Then $Y \sim NB(r, p)$. Since the pmf of X is $p(x) = p(1-p)^x$, $x = 0, 1, 2, ...$,

$$M_X(t) = \sum_{x=0}^{\infty} p[(1-p)e^t]^x = \frac{p}{1 - (1-p)e^t}, \quad t < -\log(1-p).$$

Hence, the mgf of Y is

$$M_Y(t) = [M_X(t)]^r = \frac{p^r}{[1-(1-p)e^t]^r}.$$

Let $\psi(t) = \log M_Y(t) = r \log p - r \log[1-(1-p)e^t]$. Then

$$\mu = \psi'(0) = \frac{r(1-p)e^t}{1-(1-p)e^t}\Big|_{t=0} = \frac{r(1-p)}{p}, \quad \sigma^2 = \psi''(0) = \frac{r(1-p)e^t}{[1-(1-p)e^t]^2}\Big|_{t=0} = \frac{r(1-p)}{p^2}.$$

3.1.21. Let X_1 and X_2 have a trinomial distribution. Differentiate the moment generating function to show that their covariance is $-np_1p_2$.

Solution.

By a natural extension of a binomial, the mgf of the trinomial distribution is given by

$$M_{X_1,X_2}(t_1,t_2) = [(1-p_1-p_2) + p_1e^{t_1} + p_2e^{t_2}]^n.$$

Let $\psi(t_1,t_2) = \log M_{X_1,X_2}(t_1,t_2) = n\log[(1-p_1-p_2)+p_1e^{t_1}+p_2e^{t_2}]$. Then

$$\begin{aligned}\frac{\partial\psi(t_1,t_2)}{\partial t_1} &= \frac{np_1e^{t_1}}{(1-p_1-p_2)+p_1e^{t_1}+p_2e^{t_2}},\\ \frac{\partial^2\psi(t_1,t_2)}{\partial t_1\partial t_2} &= \frac{-np_1e^{t_1}p_2e^{t_2}}{[(1-p_1-p_2)+p_1e^{t_1}+p_2e^{t_2}]^2}.\end{aligned}$$

Hence,

$$\mathrm{Cov}(X_1,X_2) = \frac{\partial^2\psi(0,0)}{\partial t_1\partial t_2} = -np_1p_2.$$

3.1.22. If a fair coin is tossed at random five independent times, find the conditional probability of five heads given that there are at least four heads.

Solution.

Let X denote the number of heads of five independent times. Then the desired possibility is given by

$$P(X=5|X\geq 4) = \frac{P(X=5, X\geq 4)}{P(X\geq 4)} = \frac{P(X=5)}{P(X=4)+P(X=5)} = \frac{(1/2)^5}{\binom{5}{4}(1/2)^5+(1/2)^5} = \frac{1}{6}.$$

3.1.25. Let

$$p(x_1,x_2) = \binom{x_1}{x_2}\left(\frac{1}{2}\right)^{x_1}\left(\frac{x_1}{15}\right), \quad \begin{array}{l} x_2 = 0,1,\ldots,x_1 \\ x_1 = 0,1,2,3,4,5, \end{array}$$

zero elsewhere, be the joint pmf of X_1 and X_2. Determine

(a) $E(X_2)$

Solution.

$$\begin{aligned}E(X_2) &= \sum_{x_1=1}^{5}\sum_{x_2=0}^{x_1} x_2\binom{x_1}{x_2}\left(\frac{1}{2}\right)^{x_1}\left(\frac{x_1}{15}\right)\\ &= \sum_{x_1=1}^{5}\left[\sum_{x_2=1}^{x_1}\binom{x_1-1}{x_2-1}\left(\frac{1}{2}\right)^{x_1-1}\right]\left(\frac{x_1}{2}\right)\left(\frac{x_1}{15}\right)\\ &= \sum_{x_1=1}^{5}\frac{x_1^2}{30} = \frac{5(6)(11)}{6(30)} = \frac{6}{11}\end{aligned}$$

since

$$\binom{x_1-1}{x_2-1}\left(\frac{1}{2}\right)^{x_1-1}, \quad x_2=1,\dots,x_1$$

is the pmf of $X_2 \sim \text{Binomial}(x_1-1, 1/2)$.

(b) $u(x_1) = E(X_2|x_1)$.

Solution.

Find $p(x_2|x_1)$ first.

$$p(x_1) = \sum_{x_2=0}^{x_1} p(x_1,x_2) = \left[\sum_{x_2=0}^{x_1}\binom{x_1}{x_2}\left(\frac{1}{2}\right)^{x_1}\right]\left(\frac{x_1}{15}\right) = \frac{x_1}{15}$$
$$\Rightarrow p(x_2|x_1) = \frac{p(x_1,x_2)}{p(x_1)} = \binom{x_1}{x_2}\left(\frac{1}{2}\right)^{x_1}.$$

Hence,

$$u(x_1) = E(X_2|x_1) = \sum_{x_2=0}^{x_1} x_2\binom{x_1}{x_2}\left(\frac{1}{2}\right)^{x_1} = \sum_{x_2=1}^{x_1}\binom{x_1-1}{x_2-1}\left(\frac{1}{2}\right)^{x_1-1}\frac{x_1}{2} = \frac{x_1}{2}, \quad x_1=1,2,3,4,5.$$

(c) $E[u(X_1)]$.

Solution.

$$E[u(X_1)] = \frac{E(X_1)}{2} = \sum_{x_1=1}^{5}\frac{x_1}{2}p(x_1) = \sum_{x_1=1}^{5}\frac{x_1^2}{30} = \frac{11}{6},$$

which is the same as that in part (a) by the iterative expectation.

3.1.26. Three fair dice are cast. In 10 independent casts, let X be the number of times all three faces are alike and let Y be the number of times only two faces are alike. Find the joint pmf of X and Y and compute $E(6XY)$.

Solution.

The joint pmf of X and Y is a trinomial distribution with $p_X = \frac{1}{36}$ and $p_Y = \frac{15}{36}$. By 3.1.21,

$$\text{Cov}(X,Y) = -np_Xp_Y = -10\frac{1}{36}\frac{15}{36} = -\frac{25}{216} = E(XY) - E(X)E(Y).$$

Hence,

$$E(XY) = \text{Cov}(X,Y) + E(X)E(Y) = -\frac{25}{216} + \frac{10}{36}\frac{10(15)}{36} = \frac{25}{24} \quad \Rightarrow \quad E(6XY) = \frac{25}{4}.$$

3.1.27. Let X have a geometric distribution. Show that

$$P(X \geq k+j | X \geq k) = P(X \geq j),$$

where k and j are nonnegative integers. Note that we sometimes say in this situation that X is **memoryless**.

Solution.

Since the pmf of X is $p(x) = p(1-p)^x$, $0 < p < 1$, $x = 0, 1, 2, ...$, the cdf is

$$F_X(x) = P(X \leq x) = \sum_{k=0}^{x} p(1-p)^k = 1-(1-p)^{x+1}.$$

Thus,

$$P(X \geq k+j | X \geq k) = \frac{P(X \geq k+j)}{P(X \geq k)} = \frac{1 - P(X \leq k+j-1)}{1 - P(X \leq k-1)} = \frac{(1-p)^{k+j}}{(1-p)^k} = (1-p)^j = P(X \geq j).$$

3.1.29. Let the independent random variables X_1 and X_2 have binomial distributions with parameters n_1, $p_1 = \frac{1}{2}$ and n_2, $p_2 = \frac{1}{2}$, respectively. Show that $Y = X_1 - X_2 + n_2$ has a binomial distribution with parameters $n = n_1 + n_2$, $p = \frac{1}{2}$.

Solution.

Since $M_{X_1}(t) = (\frac{1}{2} + \frac{e^t}{2})^{n_1}$ and $M_{X_2}(t) = (\frac{1}{2} + \frac{e^t}{2})^{n_2}$,

$$M_Y(t) = M_{X_1}(t) M_{X_2}(-t) e^{n_2 t} = \left(\frac{1}{2} + \frac{e^t}{2}\right)^{n_1} \left(\frac{1}{2} + \frac{e^{-t}}{2}\right)^{n_2} e^{n_2 t} = \left(\frac{1}{2} + \frac{e^t}{2}\right)^{n_1+n_2},$$

indicating that $Y \sim \text{Binom}(n_1 + n_2, \frac{1}{2})$.

3.1.30. Consider a shipment of 1000 items into a factory. Suppose the factory can tolerate about 5% defective items. Let X be the number of defective items in a sample without replacement of size $n = 10$. Suppose the factory returns the shipment if $X \geq 2$.

(a) Obtain the probability that the factory returns a shipment of items that has 5% defective items.

Solution. $P(X \geq 2) = 1 - P(X \leq 1) =$ `1 - phyper(1, 50, 950, 10) = 0.0853`.

(b) Suppose the shipment has 10% defective items. Obtain the probability that the factory returns such a shipment.

Solution. `1 - phyper(1, 100, 900, 10) = 0.2637`.

(c) Obtain approximations to the probabilities in parts (a) and (b) using appropriate binomial distributions.

Solution.

For part (a), `1 - pbinom(1, 10, 0.05) = 0.08613`. For (b), `1 - pbinom(1, 10, 0.1) = 0.2639`.

3.1.31. Show that the variance of a hypergeometric (N, D, n) distribution is given by expression (3.1.8). *Hint*: First obtain $E[X(X-1)]$ by proceeding in the same way as the derivation of the mean given in Section 3.1.3.

Solution.

$$E[X(X-1)] = \sum_{x=0}^{n} x(x-1) \frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}} = \frac{n(n-1)D(D-1)}{N(N-1)} \sum_{x=2}^{n} \frac{\binom{D-2}{x-2}\binom{N-D}{n-x}}{\binom{N-2}{n-2}} = \frac{n(n-1)D(D-1)}{N(N-1)}.$$

Since we have $E(X) = nD/N$,

$$\begin{aligned} \text{Var}(X) = E(X^2) - [E(X)]^2 &= E[X(X-1)] + E(X) - [E(X)]^2 \\ &= \frac{n(n-1)D(D-1)}{N(N-1)} + \frac{nD}{N} - \left(\frac{nD}{N}\right)^2 \\ &= n\frac{D}{N}\frac{N-D}{N}\frac{N-n}{N-1}. \end{aligned}$$

Note: $\text{Var}(X) \to np(1-p)$ as $N \to \infty$, where $p = D/N$ meaning that the hypergeometric approximates the binomial when N is large.

3.2 The Poisson Distribution

3.2.1. If the random variable X has a Poisson distribution such that $P(X = 1) = P(X = 2)$, find $P(X = 4)$.

Solution. $P(X = 1) = P(X = 2)$ gives the parameter $\lambda = 2$. Hence, $P(X = 4) = \frac{e^{-2}2^4}{4!} \approx 0.09$.

3.2.2. The mgf of a random variable X is $e^{4(e^t-1)}$. Show that $P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.931$.

Solution.

By the given mgf, $\lambda = \sigma^2 = 4$. Hence, $P(\mu - 2\sigma < X < \mu + 2\sigma) = P(0 < X < 8) = P(X < 8) - P(X \le 0) = P(X \le 7) - P(X = 0) =$ `ppois(7, 4) - ppois(0, 4) = 0.9305`.

3.2.3. In a lengthy manuscript, it is discovered that only 13.5 percent of the pages contain no typing errors. If we assume that the number of errors per page is a random variable with a Poisson distribution, find the percentage of pages that have exactly one error.

Solution.

Let $X \sim \text{Poisson}(\lambda)$. Then given that $P(X = 0) = 0.135 \Rightarrow e^{-\lambda} = 0.135 \Rightarrow \lambda = 2.002$. Thus,

$$P(X = 1) = \frac{e^{-2}(2)^1}{1!} = 0.270.$$

3.2.4. Let the pmf $p(x)$ be positive on and only on the nonnegative integers. Given that $p(x) = (4/x)p(x-1)$, $x = 1, 2, 3, ...$, find the formula for $p(x)$.

Solution.

$$p(x) = \frac{4}{x}p(x-1) = \cdots = \frac{4^x}{x!}p(0).$$

Also,

$$1 = \sum_{x=0}^{\infty} p(x) = p(0)\sum_{x=0}^{\infty}\frac{4^x}{x!} = p(0)e^4 \;\Rightarrow\; P(0) = e^{-4} \;\Rightarrow\; P(x) = \frac{e^{-4}4^x}{x!}.$$

That is $X \sim \text{Poisson}(4)$.

3.2.5. Let X have a Poisson distribution with $\mu = 100$. Use Chebyshev's inequality to determine a lower bound for $P(75 < X < 125)$. Next, calculate the probability using R. Is the approximation by Chebyshev's inequality accurate?

Solution.

By Chebyshev's inequality,

$$P(75 < X < 125) = P(|X - 100| < 25) = 1 - P(|X - 100| \ge 25) \ge 1 - \frac{100}{25^2} = \frac{21}{25} = 0.84.$$

Using R,

$P(75 < X < 125) =$ `ppois(124, 100) - ppois(75, 100) = 0.9858`.

So, the approximation by Chebyshev's inequality is not so accurate in this case.

3.2.10. The approximation discussed in Exercise 3.2.8 can be made precise in the following way. Suppose X_n is binomial with the parameters n and $p = \lambda/n$, for a given $\lambda > 0$. Let Y be Poisson with mean λ. Show that $P(X_n = k) \to P(Y = k)$, as $n \to \infty$, for an arbitrary but fixed value of k.

Solution.

$$\begin{aligned}
P(X_n = k) &= \frac{n!}{k!(n-k)!}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k} \\
&= \frac{\lambda^k}{k!}\frac{n(n-1)\cdots(n-k+1)}{n^k}\left(1-\frac{\lambda}{n}\right)^{-k}\left(1-\frac{\lambda}{n}\right)^{n} \\
&= \frac{\lambda^k}{k!}\frac{n}{n}\frac{n-1}{n}\cdots\frac{n-k+1}{n}\left(1-\frac{\lambda}{n}\right)^{-k}\left(1-\frac{\lambda}{n}\right)^{n} \\
&\to \frac{\lambda^k}{k!}\cdot 1^k 1^{-k} e^{-\lambda} = P(Y = k).
\end{aligned}$$

3.2.12. Compute the measures of skewness and kurtosis of the Poisson distribution with mean μ.

Solution.

Suppose X has the Poisson distribution with mean μ. Then the variance $\sigma^2 = \mu$ and the mgf is

$$M_X(t) = e^{\mu(e^t-1)} \quad \Rightarrow \quad \psi(t) = \log M_X(t) = \mu(e^t - 1).$$

Hence, the skewness is

$$\gamma = \frac{E(X-\mu)^3}{\sigma^3} = \frac{\psi^{(3)}(0)}{\mu^{1.5}} = \frac{\mu}{\mu^{1.5}} = \mu^{-0.5}.$$

Similarly, the kurtosis is

$$\kappa = \frac{E(X-\mu)^4}{\sigma^4} = \frac{\psi^{(4)}(0)}{\mu^2} = \frac{\mu}{\mu^2} = \mu^{-1}.$$

3.2.13. On the average, a grocer sells three of a certain article per week. How many of these should he have in stock so that the chance of his running out within a week is less than 0.01? Assume a Poisson distribution.

Solution.

Let $X \sim \text{Poisson}(3)$. Find the smallest x such that $P(X > x) < 0.01 \Leftrightarrow P(X \le x) > 0.99$. Since `ppois(7, 3) = 0.988` and `ppois(8, 3) = 0.996`, $x = 8$.

3.2.15. Let X have a Poisson distribution with mean 1. Compute, if it exists, the expected value $E(X!)$.

Solution.

$$E(X!) = \sum_{x=0}^{\infty} x!\frac{e^{-1}1^x}{x!} = \sum_{x=0}^{\infty} e^{-1},$$

which indicates that $E(X!)$ does not exist.

3.2.16. Let X and Y have the joint pmf $p(x, y) = e^{-2}/[x!(y-x)!]$, $y = 0, 1, 2, ...$, $x = 0, 1, ..., y$, zero elsewhere.

(a) Find the mgf $M(t_1, t_2)$ of this joint distribution.

Solution.

$$\begin{aligned}
M(t_1,t_2) &= e^{-2}\sum_{y=0}^{\infty} e^{t_2 y}\sum_{x=0}^{y}\frac{e^{t_1 x}}{[x!(y-x)!} \\
&= e^{-2}\sum_{y=0}^{\infty}\frac{e^{t_2 y}}{y!}\sum_{x=0}^{y}\binom{y}{x}e^{t_1 x}1^{y-x} \\
&= e^{-2}\sum_{y=0}^{\infty}\frac{e^{t_2 y}}{y!}[1+e^{t_1}]^y \\
&= e^{-2}\sum_{y=0}^{\infty}\frac{[e^{t_2}(1+e^{t_1})]^y}{y!} \\
&= e^{-2}\exp[(1+e^{t_1})e^{t_2}] \\
&= \exp[e^{t_2}+e^{t_1+t_2}-2]
\end{aligned}$$

(b) Compute the means, the variances, and the correlation coefficient of X and Y.

Solution.

Let $\psi(t_1,t_2) = \log M(t_1,t_2) = e^{t_2}+e^{t_1+t_2}-2$. Then

$$\begin{aligned}
\mu_X &= \frac{\partial\psi(0,0)}{\partial t_1} = 1, \quad \mu_Y = \frac{\partial\psi(0,0)}{\partial t_2} = 2 \\
\sigma_X^2 &= \frac{\partial^2\psi(0,0)}{\partial t_1^2} = 1, \quad \sigma_Y^2 = \frac{\partial^2\psi(0,0)}{\partial t_2^2} = 2, \\
\mathrm{Cov}(X,Y) &= \frac{\partial^2\psi(0,0)}{\partial t_1 t_2} = 1 \Rightarrow \rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y} = \frac{1}{\sqrt{2}}.
\end{aligned}$$

(c) Determine the conditional mean $E(X|y)$.

Solution.

Since

$$p(y) = \frac{e^{-2}}{y!}\sum_{x=0}^{y}\binom{y}{x} = \frac{e^{-2}2^y}{y!},\ y=0,1,2,... \Rightarrow Y\sim \mathrm{Poisson}(2),$$

the conditional pmf of X given $Y=y$ is given by

$$p(x|y) = \frac{p(x,y)}{p(y)} = \frac{y!}{x!(y-x)!}\frac{1}{2^y} = \frac{1}{2^y}\binom{y}{x}.$$

Hence,

$$E(X|y) = \sum_{x=0}^{y} x\frac{1}{2^y}\binom{y}{x} = \frac{y}{2^y}\sum_{x=1}^{y}\binom{y-1}{x-1} = \frac{y}{2^y}(1+1)^{y-1} = \frac{y}{2}, \quad y=0,1,2,\ldots,$$

zero elsewhere.

3.2.17. Let X_1 and X_2 be two independent random variables. Suppose that X_1 and $Y = X_1+X_2$ have Poisson distributions with means μ_1 and $\mu>\mu_1$, respectively. Find the distribution of X_2.

Solution.

Since X_1 and X_2 be two independent,

$$M_Y(t) = M_{X_1}(t)M_{X_2}(t) \Rightarrow M_{X_2}(t) = \frac{M_Y(t)}{M_{X_1}(t)} = \frac{e^{\mu(e^t-1)}}{e^{\mu_1(e^t-1)}} = e^{(\mu-\mu_1)(e^t-1)},$$

implying that $X_2\sim \mathrm{Poisson}(\mu-\mu_1)$.

3.3 The Γ, χ^2, and β Distribution

3.3.1. Supose $(1-2t)^{-6}$, $t < \frac{1}{2}$ is the mgf of the random variable X.

(a) Use R to compute $P(X < 5.23)$.

Solution.

Since $X \sim \Gamma(6, 2)$ or $X \sim \chi^2(12)$,

$$P(X < 5.23) = \texttt{pgamma(5.23, 6, scale = 2)} = \texttt{pchisq(5.23, 12)} = 0.501.$$

(b) Find the mean μ and variance σ^2 of X. Use R to compute $P(|X - \mu| < 2\sigma)$.

Solution.

Since $\mu = 6(2) = 12$ and $\sigma^2 = 6(2)^2 = 24$. Thus

$$\begin{aligned} P(|X-\mu| < 2\sigma) &= P(\mu - 2\sigma < X < \mu + 2\sigma) \\ &= P(12 - 4\sqrt{6} < X < 12 + 4\sqrt{6}) \\ &= \texttt{pchisq(12 + 4 * sqrt(6), 12) - pchisq(12 - 4 * sqrt(6), 12)} \\ &= 0.9592. \end{aligned}$$

3.3.3. Suppose the lifetime in months of an engine, working under hazardous conditions, has a Γ distribution with a mean of 10 months and a variance of 20 months squared.

(a) Determine the median lifetime of an engine.

Solution.

Since $\alpha\beta = 10$ and $\alpha\beta^2 = 20$, $\alpha = 5$ and $\beta = 2$. Hence, the median is `qgamma(0.5, 5, scale = 2)` $=$ 9.342 months.

(b) Suppose such an engine is termed successful if its lifetime exceeds 15 months. In a sample of 10 engines, determine the probability of at least 3 successful engines.

Solution.

The probability of a successful engine is $p = P(X > 15) =$ `1 - pgamma(15, 5, scale = 2)` $= 0.1321$. Let $Y \sim \text{Binomial}(10, p)$, the probability of at least 3 successful engines is

$$P(Y \geq 3) = 1 - P(Y \leq 2) = \texttt{1 - pbinom(2, 10, 0.1321)} = 0.136.$$

3.3.4. Let X be a random variable such that $E(X^m) = (m+1)!2^m$, $m = 1, 2, 3, \ldots$. Determine the mgf and the distribution of X.

Solution.

By Taylor series, the mgf of X is given by

$$M(t) = \sum_{m=0}^{\infty} \frac{M^{(m)}(0)}{m!} t^m = 1 + \sum_{m=1}^{\infty} \frac{E(X^m)}{m!} t^m = 1 + \sum_{m=1}^{\infty} (m+1)(2t)^m = \sum_{m=0}^{\infty} (m+1)(2t)^m,$$

which gives us

$$(1-2t)M(t) = \sum_{m=0}^{\infty} (2t)^m = \frac{1}{1-2t}, \quad t < \frac{1}{2}.$$

Hence, $M(t) = (1-2t)^{-2}$. So, $X \sim \Gamma(2, 2)$ or $X \sim \chi^2(4)$.

3.3.6. Let X_1, X_2, and X_3 be iid random variables, each with pdf $f(x) = e^{-x}$, $0 < x < \infty$, zero elsewhere.

(a) Find the distribution of $Y = \min(X_1, X_2, X_3)$.

Solution.

We have the cdf of X is $F_X(x) = 1 - e^{-x}$, $x > 0$. Thus, the cdf of Y is

$$\begin{aligned} F_Y(y) = P(Y \le y) = 1 - P(Y > y) &= 1 - P(X_i > y, i = 1, 2, 3) \\ &= 1 - [P(X > y)]^3 \quad \text{since } X_i's \text{ are iid} \\ &= 1 - [1 - F_X(y)]^3 \\ &= 1 - e^{-3y}. \end{aligned}$$

Hence, the pdf of Y is $f_Y(y) = 3e^{-3y}$, $y > 0$, zero elsewhere.

(b) Find the distribution of $Y = \max(X_1, X_2, X_3)$.

Solution.

Similarly,

$$\begin{aligned} F_Y(y) = P(Y \le y) &= P(X_i < y, i = 1, 2, 3) \\ &= [P(X < y)]^3 \quad \text{since } X_i's \text{ are iid} \\ &= [F_X(y)]^3 \\ &= (1 - e^{-y})^3, \quad y > 0, \end{aligned}$$

zero $y \le 0$. We do not have to show the pdf (not so simple form in this case).

3.3.7. Let X have a gamma distribution with pdf

$$f(x) = \frac{1}{\beta^2} x e^{-x/\beta}, \quad 0 < x < \infty,$$

zero elsewhere. If $x = 2$ is the unique mode of the distribution, find the parameter β and $P(X < 9.49)$.

Solution.

Solving $f'(x) = 0$, we obtain $x = \beta = 2$. Since $\alpha = 2$, $X \sim \Gamma(2, 2) = \chi^2(4)$. Hence,

$$P(X < 9.49) = \texttt{pgamma(9.49, 2, scale=2) = pchisq(9.49, 4)} = 0.950.$$

3.3.8. Compute the measures of skewness and kurtosis of a Γ distribution that has parameters α and β.

Solution.

We have $\sigma^2 = \alpha\beta^2$. Since the mgf is given by $M(t) = (1 - \beta t)^{-\alpha}$, let $\psi(t) = \log M(t) = -\alpha \log(1 - \beta t)$. Then

$$\psi'(t) = \frac{\alpha\beta}{1 - \beta t}, \quad \psi''(t) = \frac{\alpha\beta^2}{(1 - \beta t)^2}, \quad \psi^{(3)}(t) = \frac{2\alpha\beta^3}{(1 - \beta t)^3}, \quad \psi^{(4)}(t) = \frac{6\alpha\beta^4}{(1 - \beta t)^4}.$$

Hence, the measures of skewness and kurtosis are, respectively,

$$\gamma = \frac{\psi^{(3)}(0)}{\sigma^3} = \frac{2\alpha\beta^3}{(\alpha\beta^2)^{3/2}} = \frac{2}{\sqrt{\alpha}}, \quad \kappa = \frac{\psi^{(4)}(0)}{\sigma^4} = \frac{6\alpha\beta^4}{(\alpha\beta^2)^2} = \frac{6}{\alpha}.$$

3.3.10. Give a reasonable definition of a chi-square distribution with zero degrees of freedom.

Solution.

The mgf of $X \sim \chi^2(r)$ is $M_X(t) = (1 - 2t)^{-r/2}$, $t < \frac{1}{2}$. Let $r = 0$, then $M(t) = 1 \Rightarrow X = 0 \Rightarrow P(X = 0) = 1$.

3.3.15. Let X have a Poisson distribution with parameter m. If m is an experimental value of a random variable having a gamma distribution with $\alpha = 2$ and $\beta = 1$, compute $P(X = 0, 1, 2)$.

Solution.

Given that

$$f_X(x|m) = \frac{e^{-m}m^x}{x!}, \; x = 0, 1, 2, \ldots \quad f_M(m) = \frac{1}{\Gamma(2)1^2} me^{-m} = me^{-m}, \; m > 0.$$

Hence, the joint distribution of X and m and the marginal distribution of X are, respectively,

$$f_{X,m}(x, m) = f_X(x|m)f_M(m) = \frac{e^{-2m}m^{x+1}}{x!}$$

$$f_X(x) = \int_0^\infty \frac{m^{x+1}e^{-2m}}{x!} dm = \frac{1}{x!}\Gamma(x+2)\left(\frac{1}{2}\right)^{x+2} = \frac{x+1}{2^{x+2}}.$$

Hence,

$$P(X = 0, 1, 2) = f_X(0) + f_X(1) + f_X(2) = \frac{1}{4} + \frac{1}{4} + \frac{3}{16} = \frac{11}{16}.$$

3.3.16. Let X have the uniform distribution with pdf $f(x) = 1$, $0 < x < 1$, zero elsewhere. Find the cdf of $Y = -2\log X$. What is the pdf of Y?

Solution.

We have the cdf of X: $F_X(x) = x$, $0 < x < 1$. Hence, the cdf of Y is

$$F_Y(y) = P(-2\log X \le y) = P(X \ge e^{-y/2}) = 1 - F(e^{-y/2}) = 1 - e^{-y/2}, \quad 0 < y < \infty,$$

which gives the pdf of Y: $f_Y(y) = \frac{1}{2}e^{-y/2}$, $y > 0$, zero elsewhere. That is $Y \sim \Gamma(1, 2) = \chi^2(2)$.

3.3.23. Let X_1 and X_2 be independent random variables. Let X_1 and $Y = X_1 + X_2$ have chi-square distributions with r_1 and r degrees of freedom, respectively. Here $r_1 < r$. Show that X_2 has a chi-square distribution with $r - r_1$ degrees of freedom.

Solution.

Since X_1 and X_2 are independent,

$$M_Y(t) = M_{X_1}(t)M_{X_2}(t) \;\Rightarrow\; M_{X_2}(t) = \frac{M_Y(t)}{M_{X_1}(t)} = \frac{(1-2t)^{-r/2}}{(1-2t)^{-r_1/2}} = (1-2t)^{-(r-r_1)/2},$$

which gives $X_2 \sim \chi^2(r - r_1)$.

3.3.24. Let X_1, X_2 be two independent random variables having gamma distributions with parameters $\alpha_1 = 3$, $\beta_1 = 3$ and $\alpha_2 = 5$, $\beta_2 = 1$, respectively.

(a) Find the mgf of $Y = 2X_1 + 6X_2$.

Solution. Since $X_1 \perp X_2$, $M_Y(t) = M_{X_1}(2t)M_{X_2}(6t) = [1 - 3(2t)]^{-3}(1 - 6t)^{-5} = (1 - 6t)^{-8}$, $t < \frac{1}{6}$.

(b) What is the distribution of Y?

Solution. $Y \sim \Gamma(8, 6)$.

3.3.26. Let X denote time until failure of a device and let $r(x)$ denote the hazard function of X.

(a) If $r(x) = cx^b$, where c and b are positive constants, show that X has a **Weibull** distribution; i.e.,

$$f(x) = \begin{cases} cx^b \exp\left\{-\frac{cx^{b+1}}{b+1}\right\} & 0 < x < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

Solution.

Since $r(x) = -(d/dx)\log[1 - F(x)]$, $F(x) = 1 - e^{-\int_0^x r(u)du}$ and $f(x) = r(x)e^{-\int_0^x r(u)du}$. Hence,

$$\int_0^x r(u)du = \frac{cx^{b+1}}{b+1} \quad \Rightarrow \quad f(x) = cx^b e^{-\frac{cx^{b+1}}{b+1}}, \quad 0 < x < \infty.$$

(b) If $r(x) = ce^{bx}$, where c and b are positive constants, show that X has a **Gompertz** cdf given by

$$F(x) = \begin{cases} 1 - \exp\left\{\frac{c}{b}(1 - e^{bx})\right\} & 0 < x < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

This is frequently used by actuaries as a distribution of the length of human life.

Solution.

$$\int_0^x r(u)du = -\frac{c}{b}(1 - e^{bx}) \quad \Rightarrow \quad F(x) = 1 - e^{\frac{c}{b}(1-e^{bx})}, \quad 0 < x < \infty.$$

(c) If $r(x) = bx$, linear hazard rate, show that the pdf of X is

$$f(x) = \begin{cases} bxe^{-bx^2/2} & 0 < x < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

This pdf is called the **Rayleigh** pdf.

Solution.

$$\int_0^x r(u)du = \frac{bx^2}{2} \quad \Rightarrow \quad f(x) = bxe^{-bx^2/2}, \quad 0 < x < \infty.$$

3.4 The Normal Distribution

3.4.1. If

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw,$$

show that $\Phi(-z) = 1 - \Phi(z)$.

Solution.

$$\Phi(-z) = \int_{-\infty}^{-z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw = 1 - \int_{-z}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw = 1 - \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-(w')^2/2} dw' = 1 - \Phi(z)$$

where $w' = -w$ and so $dw' = -dw$.

3.4.2. If X is $N(75, 100)$, find $P(X < 60)$ and $P(70 < X < 100)$ by using either Table II or the R command `pnorm`.

Solution.

$$\begin{aligned} P(X < 60) &= P\left(\frac{X - 75}{10} < -1.5\right) = \Phi(-1.5) = 1 - \Phi(1.5) = 1 - 0.9332 = 0.0668, \\ &= \texttt{pnorm(60, 75, 10)} = 0.06681, \\ P(70 < X < 100) &= \Phi(2.5) - \Phi(-0.5) = 0.9938 - (1 - 0.6915) = 0.6853, \\ &= \texttt{pnorm(100, 75, 10) - pnorm(70, 75, 10)} = 0.68525. \end{aligned}$$

3.4.3. If X is $N(\mu, \sigma^2)$, find b so that $P[-b < (X - \mu)/\sigma < b] = 0.90$, by using either Table II of Appendix D or the R command `qnorm`.

Solution. $b = 1.645$.

3.4.5. Show that the constant c can be selected so that $f(x) = c2^{-x^2}$, $-\infty < x < \infty$, satisfies the conditions of a normal pdf.

Solution.

Since $2^{-x^2} = e^{-x^2 \log 2} = e^{x^2/(1/\log 2)}$, consider $X \sim N(0, \frac{1}{2\log 2})$. Then the pdf of X is

$$f(x) = \frac{1}{\sqrt{2\pi \frac{1}{2\log 2}}} e^{-x^2 \log 2} = \sqrt{\frac{\log 2}{\pi}} e^{-x^2 \log 2}, \quad -\infty < x < \infty.$$

Hence, $c = \sqrt{\frac{\log 2}{\pi}}$.

3.4.6. If X is $N(\mu, \sigma^2)$, show that $E(|X - \mu|) = \sigma\sqrt{2/\pi}$.

Solution.

WLOG, $\mu = 0$. Because of the symmetry of a normal pdf,

$$E(|X|) = 2\int_0^\infty x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/(2\sigma^2)} dx = \frac{2}{\sqrt{2\pi\sigma^2}} \left[-\sigma^2 e^{-x^2/(2\sigma^2)}\right]_0^\infty = \frac{2\sigma^2}{\sqrt{2\pi\sigma^2}} = \sigma\sqrt{\frac{2}{\pi}}.$$

3.4.8. Evaluate $\int_2^3 \exp[-2(x-3)^2]dx$.

Solution.

Suppose $X \sim N(3, 1/4)$, the pdf of X is

$$f(x) = \sqrt{\frac{2}{\pi}} e^{-2(x-3)^2}.$$

Hence,

$$\int_2^3 \sqrt{\frac{2}{\pi}} e^{-2(x-3)^2} dx = P(X \le 3) - P(X \le 2) = \Phi(0) - \Phi(-2) = \frac{1}{2} - \Phi(-2)$$
$$\Rightarrow \int_2^3 \exp[-2(x-3)^2]dx = \sqrt{\frac{\pi}{2}} \left[\frac{1}{2} - \Phi(-2)\right]$$

3.4.10. If e^{3t+8t^2} is the mgf of the random variable X, find $P(-1 < X < 9)$.

Solution.

By the mgf, we have $X \sim N(3, 4^2)$. Hence,

$$P(-1 < X < 9) = P(-1 < Z < 1.5) = 0.7745,$$
$$= \texttt{pnorm(9, 3, 4) - pnorm(-1, 3, 4)} = 0.77454.$$

3.4.11. Let the random variable X have the pdf

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}, \quad 0 < x < \infty, \quad \text{zero elsewhere.}$$

(a) Find the mean and the variance of X.

Solution.

$$E(X) = \int_0^\infty x \frac{2}{\sqrt{2\pi}} e^{-x^2/2} dx = \sqrt{\frac{2}{\pi}} - e^{-x^2/2}\Big|_0^\infty = \sqrt{\frac{2}{\pi}},$$
$$E(X^2) = \int_0^\infty x^2 \frac{2}{\sqrt{2\pi}} e^{-x^2/2} dx = \cdots = \int_0^\infty \frac{2}{\sqrt{2\pi}} e^{-x^2/2} dx = 1,$$
$$\Rightarrow \text{Var}(X) = E(X^2) - E(X)^2 = 1 - \frac{2}{\pi}.$$

(b) Find the cdf and hazard function of X.

Solution.

$$\begin{aligned} F_X(x) &= 2\int_0^x \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \\ &= 2\left(\int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du - \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du\right) \\ &= 2[\Phi(x) - 0.5] = 2\Phi(x) - 1. \end{aligned}$$

Also, let $\gamma(x)$ denote the hazard function of X, then

$$\gamma(x) = \frac{f(x)}{1 - F_X(x)} = \frac{f(x)}{2[1 - \Phi(x)]}.$$

3.4.12. Let X be $N(5, 10)$. Find $P[0.04 < (X - 5)^2 < 38.4]$.

Solution.

$$\frac{X-5}{\sqrt{10}} \sim N(0,1) \;\Rightarrow\; \frac{(X-5)^2}{10} \sim \chi^2(1).$$

Hence,

$$\begin{aligned} P[0.04 < (X-5)^2 < 38.4] &= P\left[0.004 < \frac{(X-5)^2}{10} < 3.84\right] \\ &= \texttt{pchisq(3.84, 1) - pchisq(0.004, 1)} = 0.900. \end{aligned}$$

3.4.13. If X is $N(1, 4)$, compute the probability $P(1 < X^2 < 9)$.

Solution.

$$\begin{aligned} P(1 < X^2 < 9) &= P(-3 < X < -1) + P(1 < X < 3) \\ &= P(-2 < Z < -1) + P(0 < Z < 1) \\ &= \texttt{pnorm(-1) - pnorm(-2) + pnorm(1) - pnorm(0)} \\ &= 0.4772. \end{aligned}$$

3.4.15. Let X be a random variable such that $E(X^{2m}) = (2m)!/(2^m m!)$, $m = 1, 2, 3, ...$ and $E(X^{2m-1}) = 0$, $m = 1, 2, 3, ...$ Find the mgf and the pdf of X.

Solution.

$$M_X(t) = \sum_{k=0}^{\infty} \frac{M^{(k)}(0)}{k!} t^k = \sum_{k=0}^{\infty} \frac{E(X^k)}{k!} t^k = \sum_{m=0}^{\infty} \frac{E(X^{2m})}{(2m)!} t^{2m} + \sum_{m=1}^{\infty} \frac{E(X^{2m-1})}{(2m-1)!} t^{2m-1} = \sum_{m=0}^{\infty} \frac{(\frac{t^2}{2})^m}{m!} = e^{\frac{t^2}{2}}.$$

Hence, $X \sim N(0, 1)$.

3.4.16. Let the mutually independent random variables X_1, X_2, and X_3 be $N(0, 1)$, $N(2, 4)$, and $N(-1, 1)$, respectively. Compute the probability that exactly two of these three variables are less than zero.

Solution.

We have $P(X_1 < 0) = 0.5$. Let $P(X_2 < 0) = \Phi(-1) = a = 0.1587$, then $P(X_3 < 1) = \Phi(1) = 1 - a$. The desired probability is given by

$$\begin{aligned} &P(X_1 < 0)P(X_2 < 0)P(X_3 \geq 0) + P(X_1 < 0)P(X_2 \geq 0)P(X_3 < 0) + P(X_1 \geq 0)P(X_2 < 0)P(X_3 < 0) \\ &\qquad = 0.5a^2 + 0.5(1-a)^2 + 0.5a(1-a) = 0.5(a^2 - a + 1) = 0.433. \end{aligned}$$

3.4.17. Compute the measures of skewness and kurtosis of a distribution which is $N(\mu, \sigma^2)$. See Exercises 1.9.14 and 1.9.15 for the definitions of skewness and kurtosis, respectively.

Solution.

Let γ and κ denote the skewness and kurtosis, respectively and $Z \sim N(0,1)$. Then

$$\gamma = \frac{E(X-\mu)^3}{\sigma^3} = E(Z^3) = \int_{-\infty}^{\infty} z^3 f(z)dz = \int_0^{\infty} z^3 f(z)dz + \int_{-\infty}^{0} z^3 f(z)dz = 0$$

because $f(-z) = f(z)$. Next,

$$\kappa = \frac{E(X-\mu)^4}{\sigma^4} = E(Z^4) = \text{Var}(Z^2) + [E(Z^2)]^2 = 2 + 1^2 = 3$$

because $Z^2 \sim \chi^2(1)$.

3.4.19. Let the random variable X be $N(\mu, \sigma^2)$. What would this distribution be if $\sigma^2 = 0$?

Solution.

If $\sigma^2 = 0$, the mgf of X will be $M(t) = e^{\mu t} \Rightarrow N(\mu, 0)$. So X is degenerate at μ, or $P(X = \mu) = 1$.

3.4.20. Let Y have a **truncated** distribution with pdf $g(y) = \phi(y)/[\Phi(b) - \Phi(a)]$, for $a < y < b$, zero elsewhere, where $\phi(x)$ and $\Phi(x)$ are, respectively, the pdf and distribution function of a standard normal distribution. Show then that $E(Y)$ is equal to $[\phi(a) - \phi(b)]/[\Phi(b) - \Phi(a)]$.

Solution.

$$E(Y) = \frac{\int_a^b y\phi(y)dy}{\Phi(b) - \Phi(a)} = \frac{\int_a^b y \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy}{\Phi(b) - \Phi(a)} = \frac{\left[-\frac{e^{-y^2/2}}{\sqrt{2\pi}}\right]_a^b}{\Phi(b) - \Phi(a)} = \frac{\phi(a) - \phi(b)}{\Phi(b) - \Phi(a)}.$$

3.4.22. Let X and Y be independent random variables, each with a distribution that is $N(0,1)$. Let $Z = X + Y$. Find the integral that represents the cdf $G(z) = P(X + Y \le z)$ of Z. Determine the pdf of Z.

Solution.

Since X and Y are independent, the joint pdf of the two r.v.s is

$$f(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}, \quad -\infty < x, y < \infty.$$

Hence,

$$\begin{aligned}
G(z) &= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} \frac{1}{2\pi} e^{-(x^2+y^2)/2} dy dx \\
\Rightarrow G'(z) &= \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial z} \int_{-\infty}^{z-x} \frac{1}{2\pi} e^{-(x^2+y^2)/2} dy\right] dx \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-(x^2+(z-x)^2)/2} dx \\
&= \frac{1}{\sqrt{2\pi(2)}} e^{-z^2/4} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1/2)}} e^{-(x-\frac{z}{2})^2} dx \\
&= \frac{1}{\sqrt{4\pi}} e^{-z^2/4},
\end{aligned}$$

which gives $Z \sim N(0,2)$.

3.4.29. Let X_1 and X_2 be independent with normal distributions $N(6,1)$ and $N(7,1)$, respectively. Find $P(X_1 > X_2)$.

Solution.

Since $X_1 - X_2 \sim N(-1, 2)$,

$$P(X_1 > X_2) = P(X_1 - X_2 > 0) = P\left(\frac{(X_1 - X_2) - (-1)}{\sqrt{2}} > \frac{1}{\sqrt{2}}\right) = 1 - \Phi(1/\sqrt{2}) = 0.240.$$

3.4.30. Compute $P(X_1 + 2X_2 - 2X_3 > 7)$ if X_1, X_2, X_3 are iid with common distribution $N(1, 4)$.

Solution.

Let $Y = X_1 + 2X_2 - 2X_3$. Then

$$\mu_Y = E(X_1 + 2X_2 - 2X_3) = 1 + 2 - 2 = 1,$$
$$\sigma_Y^2 = \mathrm{Var}(X_1 + 2X_2 - 2X_3) = \mathrm{Var}(X_1) + 4\mathrm{Var}(X_2) + 4\mathrm{Var}(X_3) = 36,$$

so $Y \sim N(1, 6^2)$. Hence, $P(Y > 7) = P(Z > 1) = 0.1586$.

3.4.31. A certain job is completed in three steps in series. The means and standard deviations for the steps are (in minutes)

Step	Mean	Standard Deviation
1	17	2
2	13	1
3	13	2

Assuming independent steps and normal distributions, compute the probability that the job takes less than 40 minutes to complete.

Solution.

Since $X_1 + X_2 + X_3 \sim N(43, 9)$,

$$P(X_1 + X_2 + X_3 < 40) = P\left[\frac{(X_1 + X_2 + X_3) - 43}{3} < -1\right] = \Phi(-1) = 0.1586.$$

3.4.32. Let X be $N(0, 1)$. Use the moment generating function technique to show that $Y = X^2$ is $\chi^2(1)$.

Solution.

$$\begin{aligned} M_Y(t) = E(e^{tX^2}) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(1-2t)x^2/2} dx \\ &= (1-2t)^{-1/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw \quad (w = x\sqrt{1-2t}) \\ &= (1-2t)^{-1/2}, \end{aligned}$$

meaning that $Y \sim \Gamma(1/2, 2) = \chi^2(1)$.

3.4.33. Suppose X_1, X_2 are iid with a common standard normal distribution. Find the joint pdf of $Y_1 = X_1^2 + X_2^2$ and $Y_2 = X_2$ and the marginal pdf of Y_1.

Solution.

The joint pdf of X_1 and X_2 is

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi} e^{-(x_1^2 + x_2^2)/2}, \quad -\infty < x_1 < \infty,\ -\infty < x_2 < \infty.$$

The inverse functions are $x_1 = \pm\sqrt{y_1 - y_2^2}$ and $x_2 = y_2$ and then the Jacobian is $J = (2\sqrt{y_1 - y_2^2})^{-1}$. Hence,

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= f_{X_1,X_2}(\sqrt{y_1 - y_2^2}, y_2)|J| + f_{X_1,X_2}(-\sqrt{y_1 - y_2^2}, y_2)|J| \\ &= \frac{1}{2\pi\sqrt{y_1 - y_2^2}} e^{-y_1/2}, \quad -\sqrt{y_1} < y_2 < \sqrt{y_1},\ 0 < y_1 < \infty \end{aligned}$$

and the marginal pdf of Y_1 is

$$f_{Y_1}(y_1) = \frac{e^{-y_1/2}}{2\pi} \int_{-\sqrt{y_1}}^{\sqrt{y_1}} \frac{dy_2}{\sqrt{y_1 - y_2^2}} = \cdots = \frac{e^{-y_1/2}}{2}$$

by transforming $y_2 = \sqrt{y_1}\cos\theta,\ 0 < \theta < \pi$. Thus, $Y_1 \sim \Gamma(1, 2) = \chi^2(2)$.

3.5 The Multivariate Normal Distribution

3.5.1. Let X and Y have a bivariate normal distribution with respective parameters $\mu_x = 2.8$, $\mu_y = 110$, $\sigma_x^2 = 0.16$, $\sigma_y^2 = 100$, and $\rho = 0.6$. Using R, compute:

(a) $P(106 < Y < 124)$.

Solution.

$Y \sim N(110, 10^2)$, so $P(106 < Y < 124) = P(-0.4 < Z < 1.4) =$ `pnorm(1.4) - pnorm(-0.4)` $= 0.575$.

(b) $P(106 < Y < 124|X = 3.2)$.

Solution.

$Y|X = 3.2$ is normally distributed with the mean and variance:

$$\begin{aligned} E(Y|X = 3.2) &= \mu_y + \rho\frac{\sigma_y}{\sigma_x}(x - \mu_x) = 110 + 0.6\frac{10}{0.4}(3.2 - 2.8) = 116, \\ \text{Var}(Y|X = 3.2) &= \sigma_y^2(1 - \rho^2) = 100(1 - 0.6^2) = 64 = 8^2. \end{aligned}$$

Hence,

$$\begin{aligned} P(106 < Y < 124|X = 3.2) &= P\left(-1.25 < \frac{Y - 116}{8} < 1.0\right) \\ &= \texttt{pnorm(1) - pnorm(-1.25)} \\ &= 0.736. \end{aligned}$$

3.5.2. Let X and Y have a bivariate normal distribution with parameters $\mu_1 = 3$, $\mu_2 = 1$, $\sigma_1^2 = 16$, $\sigma_2^2 = 25$, and $\rho = \frac{3}{5}$. Using R, determine the following probabilities:

(a) $P(3 < Y < 8)$.

Solution.

$Y \sim N(1, 5^2)$, so $P(3 < Y < 8) = P(0.4 < Z < 1.4) =$ `pnorm(1.4) - pnorm(0.4)` $= 0.264$.

(b) $P(3 < Y < 8|X = 7)$.

Solution.

$Y|X = 7$ is normally distributed with the mean and variance:

$$\begin{aligned} E(Y|X = 7) &= \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1) = 1 + 0.6\frac{5}{4}(7 - 3) = 4, \\ \text{Var}(Y|X = 7) &= \sigma_2^2(1 - \rho^2) = 25(1 - (3/5)^2) = 16 = 4^2. \end{aligned}$$

Hence,

$$\begin{aligned} P(3 < Y < 8 | X = 7) &= P\left(-0.25 < \frac{Y-4}{4} < 1.0\right) \\ &= \texttt{pnorm(1) - pnorm(-0.25)} \\ &= 0.440. \end{aligned}$$

(c) $P(-3 < X < 3)$.

Solution.

$X \sim N(3, 4^2)$, so $P(-3 < X < 3) = P(-1.5 < Z < 0) =$ `pnorm(0) - pnorm(-1.5)` $= 0.433$.

(d) $P(-3 < X < 3 | Y = -4)$.

Solution.

$X|Y = -4$ is normally distributed with the mean and variance:

$$\begin{aligned} E(X|Y = -4) &= \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y - \mu_2) = 3 + 0.6\frac{4}{5}(-4 - 1) = 0.6, \\ \mathrm{Var}(X|Y = -4) &= \sigma_1^2(1 - \rho^2) = 16(1 - (3/5)^2) = (16/5)^2. \end{aligned}$$

Hence,

$$\begin{aligned} P(-3 < X < 3 | Y = -4) &= P\left(-\frac{9}{8} < \frac{X - 0.6}{3.2} < \frac{3}{4}\right) \\ &= \texttt{pnorm(3/4) - pnorm(-9/8)} \\ &= 0.643. \end{aligned}$$

3.5.6. Let U and V be independent random variables, each having a standard normal distribution. Show that the mgf $E(e^{t(UV)})$ of the random variable UV is $(1 - t^2)^{-1/2}$, $-1 < t < 1$.

Solution.

Using iterative expectation, we obtain $E(e^{tUV}) = E_V[E_U(e^{tUV}|V)]$. First, consider $V = v$ (fixed):

$$E[e^{t(UV)}|V = v] = E[e^{(tv)U}] = M_U(vt) = e^{\frac{v^2t^2}{2}}.$$

Hence,

$$E(e^{tUV}) = E_V[E_U(e^{tUV}|V)] = E(e^{\frac{t^2V^2}{2}}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(1-t^2)v^2/2} dv = (1 - t^2)^{-1/2}, \quad -1 < t < 1.$$

3.5.11. Let X, Y, and Z have the joint pdf

$$\left(\frac{1}{2\pi}\right)^{3/2} \exp\left(-\frac{x^2 + y^2 + z^2}{2}\right)\left[1 + xyz \exp\left(-\frac{x^2 + y^2 + z^2}{2}\right)\right],$$

where $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$, While X, Y, and Z are obviously dependent, show that X, Y , and Z are pairwise independent and that each pair has a bivariate normal distribution.

Solution.

The joint pdf of X and Y is given by

$$
\begin{aligned}
f_{X,Y}(x,y) &= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\right)^{3/2} \exp\left(-\frac{x^2+y^2+z^2}{2}\right)\left[1 + xyz\exp\left(-\frac{x^2+y^2+z^2}{2}\right)\right] dz \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\right)^{3/2} \exp\left(-\frac{x^2+y^2+z^2}{2}\right) dz + \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\right)^{3/2} xyz\exp\left[-(x^2+y^2+z^2)\right] dz \\
&= \left(\frac{1}{2\pi}\right)\exp\left(-\frac{x^2+y^2}{2}\right)\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz - \left(\frac{1}{2\pi}\right)^{3/2}\frac{xy\exp\left[-(x^2+y^2+z^2)\right]}{2}\Big|_{-\infty}^{\infty} \\
&= \left(\frac{1}{2\pi}\right)\exp\left(-\frac{x^2+y^2}{2}\right) - 0 \\
&= \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\frac{1}{\sqrt{2\pi}}e^{-y^2/2},
\end{aligned}
$$

which gives the desired result.

3.5.12. Let X and Y have a bivariate normal distribution with parameters $\mu_1 = \mu_2 = 0$, $\sigma_1^2 = \sigma_2^2 = 1$, and correlation coefficient ρ. Find the distribution of the random variable $Z = aX + bY$ in which a and b are nonzero constants.

Solution.

Since Z is written as

$$Z = \begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \mathbf{AX},$$

by Theorem 3.5.2, $Z \sim N_1(\mathbf{A}\boldsymbol{\mu}, \mathbf{A\Sigma A}')$, where

$$
\begin{aligned}
\mathbf{A}\boldsymbol{\mu} &= \begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0, \\
\mathbf{A\Sigma A}' &= \begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \\
&= \begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \\
&= (a^2+b^2)(1+\rho).
\end{aligned}
$$

Thus, $Z \sim N(0, (a^2+b^2)(1+\rho))$.

3.5.16. Suppose $\mathbf{X}$ is distributed $N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Determine the distribution of the random vector (X_1+X_2, X_1-X_2). Show that X_1+X_2 and X_1-X_2 are independent if $\text{Var}(X_1) = \text{Var}(X_2)$.

Solution.

Since $\mathbf{Y} \equiv (X_1+X_2, X_1-X_2)'$ is written as

$$\mathbf{Y} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \mathbf{AX},$$

by Theorem 3.5.2, $\mathbf{Y} \sim N_2(\mathbf{A}\boldsymbol{\mu}, \mathbf{A\Sigma A}')$, where the variance is

$$
\begin{aligned}
\mathbf{A\Sigma A}' &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} \sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2 & \sigma_1^2 - \sigma_2^2 \\ \sigma_1^2 - \sigma_2^2 & \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2 \end{bmatrix}.
\end{aligned}
$$

Hence, if $\sigma_1^2 = \sigma_2^2$ or $\text{Var}(X_1) = \text{Var}(X_2) = \sigma^2$, then

$$\mathbf{A\Sigma A}' = \begin{bmatrix} 2\sigma^2(1+\rho) & 0 \\ 0 & 2\sigma^2(1-\rho) \end{bmatrix},$$

indicating that $X_1 + X_2 \sim N(\mu_1 + \mu_2, 2\sigma^2(1+\rho))$ and $X_1 - X_2 \sim N(\mu_1 - \mu_2, 2\sigma^2(1-\rho))$ are independent.

3.5.22. Readers may have encountered the multiple regression model in a previous course in statistics. We can briefly write it as follows. Suppose we have a vector of n observations Y which has the distribution $N_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{I})$, where $\mathbf{X}$ is an $n \times p$ matrix of known values, which has full column rank p, and $\boldsymbol{\beta}$ is a $p \times 1$ vector of unknown parameters. The least squares estimator of $\boldsymbol{\beta}$ is

$$\widehat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}.$$

(a) Determine the distribution of $\widehat{\boldsymbol{\beta}}$.

Solution.

Since $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ is fixed, by the theorem 3.5.2, $\widehat{\boldsymbol{\beta}}$ has a normal distribution with the mean and variance, respectively:

$$\begin{aligned} E(\widehat{\boldsymbol{\beta}}) &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'E(\mathbf{Y}) = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\boldsymbol{\beta} = \boldsymbol{\beta}, \\ \mathrm{Var}(\widehat{\boldsymbol{\beta}}) &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathrm{Var}(Y)\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}. \end{aligned}$$

(b) Let $\widehat{\mathbf{Y}} = \mathbf{X}\widehat{\boldsymbol{\beta}}$. Determine the distribution of $\widehat{\mathbf{Y}}$.

Solution.

As with part (a), $\widehat{\mathbf{Y}}$ is also normally distributed with

$$\begin{aligned} \mu &= \mathbf{X}E(\widehat{\boldsymbol{\beta}}) = \mathbf{X}\boldsymbol{\beta}, \\ \sigma^2 &= \mathbf{X}\mathrm{Var}(\boldsymbol{\beta})\mathbf{X}' = \sigma^2\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'. \end{aligned}$$

(c) Let $\widehat{\mathbf{e}} = \mathbf{Y} - \widehat{\mathbf{Y}}$. Determine the distribution of $\widehat{\mathbf{e}}$.

Solution.

By part (b), we see that $\widehat{\mathbf{e}}$ also follows a normal distribution with

$$\begin{aligned} \mu &= E(\mathbf{Y}) - E(\widehat{\mathbf{Y}}) = \mathbf{0}, \\ \sigma^2 &= \mathrm{Var}(\mathbf{Y}) + \mathrm{Var}(\widehat{\mathbf{Y}}) = \sigma^2(\mathbf{I} + \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}') \end{aligned}$$

since $\mathbf{Y}$ and $\widehat{\mathbf{Y}}$ are independent.

(d) By writing the random vector $(\widehat{\mathbf{Y}}', \widehat{\mathbf{e}}')'$ as a linear function of $\mathbf{Y}$, show that the random vectors $\widehat{\mathbf{Y}}$ and $\widehat{\mathbf{e}}$ are independent.

Solution.

$$\mathbf{Z} = \begin{bmatrix} \widehat{\mathbf{Y}} \\ \widehat{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{Y} - \widehat{\mathbf{e}} \\ \widehat{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{1}_n' \\ \mathbf{0}_n' \end{bmatrix} \mathbf{Y} - \begin{bmatrix} \mathbf{1}_n' \\ -\mathbf{1}_n' \end{bmatrix} \widehat{\mathbf{e}}.$$

Hence, by the theorem 3.5.2, the variance-covariance matrix is

$$\begin{bmatrix} \mathbf{1}_n' \\ \mathbf{0}_n' \end{bmatrix} \mathrm{Var}(\mathbf{Y}) \begin{bmatrix} \mathbf{1}_n & \mathbf{0}_n \end{bmatrix} = \sigma^2 \begin{bmatrix} n & 0 \\ 0 & 0 \end{bmatrix},$$

which implies that $\widehat{\mathbf{Y}}$ and $\widehat{\mathbf{e}}$ are independent because the covariances are zero.

(e) Show that $\widehat{\boldsymbol{\beta}}$ solves the least squares problem; that is,

$$||\mathbf{Y} - \mathbf{X}\widehat{\boldsymbol{\beta}}||^2 \min_{\mathbf{b} \in R^p} ||\mathbf{Y} - \mathbf{X}\mathbf{b}||^2.$$

Solution.

$$\begin{aligned}||\mathbf{Y}-\mathbf{X}\widehat{\boldsymbol{\beta}}||^2 &= (\mathbf{Y}-\mathbf{X}\widehat{\boldsymbol{\beta}})'(\mathbf{Y}-\mathbf{X}\widehat{\boldsymbol{\beta}})\\ &= ||\mathbf{Y}||^2 - 2\mathbf{Y}'\mathbf{X}\widehat{\boldsymbol{\beta}} + \widehat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{X}\widehat{\boldsymbol{\beta}}\end{aligned}$$

Then, the derivative of this with respect to $\boldsymbol{\beta}$ is

$$\frac{\partial}{\partial\boldsymbol{\beta}}||\mathbf{Y}-\mathbf{X}\widehat{\boldsymbol{\beta}}||^2 = \mathbf{0} - 2\mathbf{X}'\mathbf{Y} + 2\mathbf{X}'\mathbf{X}\widehat{\boldsymbol{\beta}}.$$

Solving that this equals zero, we obtain $\mathbf{X}'\mathbf{X}\widehat{\boldsymbol{\beta}} = \mathbf{X}'\mathbf{Y}$. Given that $\mathbf{X}$ is full rank (nonsingular), the inverse of $\mathbf{X}'\mathbf{X}$ exists. Therefore, $\widehat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$.

3.6. t- and F-Distributions

3.6.1. Let T have a t-distribution with 10 degrees of freedom. Find $P(|T| > 2.228)$ from either Table III or by using R.

Solution. Since t-distribution is symmetric and `pt(-2.228, 10)` $= 0.025$, $P(|T| > 2.228) = 0.05$.

3.6.2. Let T have a t-distribution with 14 degrees of freedom. Determine b so that $P(-b < T < b) = 0.90$. Use either Table III or by using R.

Solution. Since t-distribution is symmetric, find $P(T > b) = 0.05$. $b =$ `qt(0.95, 14)` $= 1.761$.

3.6.6. In expression (3.4.13), the normal location model was presented. Often real data, though, have more outliers than the normal distribution allows. Based on Exercise 3.6.5, outliers are more probable for t-distributions with small degrees of freedom. Consider a location model of the form

$$X = \mu + e,$$

where e has a t-distribution with 3 degrees of freedom. Determine the standard deviation σ of X and then find $P(|X-\mu| \geq \sigma)$.

Solution.

$$\sigma^2 = \text{Var}(e) = \frac{r}{r-2} = 3 \;\Rightarrow\; \sigma = \sqrt{3}.$$

Hence, $P(|X-\mu| \geq \sigma) = P(|e| \geq \sqrt{3}) =$ `2 * pt(-sqrt(3), 3)` $= 0.1817$.

3.6.9. Let F have an F-distribution with parameters r_1 and r_2. Argue that $1/F$ has an F-distribution with parameters r_2 and r_1.

Solution.

Let $U \sim \chi^2(r_1)$ and $V \sim \chi^2(r_2)$,

$$F = \frac{U/r_1}{V/r_2} \sim F(r_1, r_2) \;\Rightarrow\; \frac{1}{F} = \frac{V/r_2}{U/r_1} \sim F(r_2, r_1),$$

which is the desired result.

3.6.10. Suppose F has an F-distribution with parameters $r_1 = 5$ and $r_2 = 10$. Using only 95th percentiles of F-distributions, find a and b so that$P(F \leq a) = 0.05$ and $P(F \leq b) = 0.95$, and, accordingly, $P(a < F < b) = 0.90$.

Solution. $a =$ `qf(0.05, 5, 10)` $= 0.211$ and $b =$ `qf(0.95, 5, 10)` $= 3.326$.

3.6.11. Let $T = W/\sqrt{V/r}$, where the independent variables W and V are, respectively, normal with mean zero and variance 1 and chi-square with r degrees of freedom. Show that T^2 has an F-distribution with parameters $r_1 = 1$ and $r_2 = r$.

Solution.

Since $W^2 \sim \chi^2(1)$,

$$T^2 = \frac{W^2/1}{V/r} \sim F(1, r).$$

3.6.12. Show that the t-distribution with $r = 1$ degree of freedom and the Cauchy distribution are the same.

Solution.

Substituting $r = 1$ to the pdf of T:

$$\begin{aligned} f(t) &= \frac{\Gamma[(r+1)/2]}{\sqrt{\pi r}\Gamma(r/2)} \frac{1}{(1+t^2/r)^{(r+1)/2}} \\ &= \frac{\Gamma(1)}{\sqrt{\pi}\Gamma(1/2)} \frac{1}{(1+t^2)} \\ &= \frac{1}{\pi(1+t^2)} \quad \text{since } \Gamma(1/2) = \sqrt{\pi}, \end{aligned}$$

provided $-\infty < t < \infty$. This is a pdf of the Cauchy distribution.

3.6.14. Show that

$$Y = \frac{1}{1 + (r_1/r_2)W}$$

where W has an F-distribution with parameters r_1 and r_2, has a beta distribution.

Solution.

Let $U \sim \chi^2(r_1) = \Gamma(r_1/2, 2)$ and $V \sim \chi^2(r_2) = \Gamma(r_2/2, 2)$, then Since $W = (U/r_1)/(V/r_2)$,

$$Y = \frac{1}{1 + U/V} = \frac{V}{V + U},$$

indicating $Y \sim \text{Beta}(r_2/2, r_1/2)$.

3.6.15. Let X_1, X_2 be iid with common distribution having the pdf $f(x) = e^{-x}$, $0 < x < \infty$, zero elsewhere. Show that $Z = X_1/X_2$ has an F-distribution.

Solution.

Since $X_i \sim \Gamma(1, 1)$, let $Y_i = 2X_i$, $i = 1, 2$, then the mgf of Y is

$$M_{Y_i}(t) = M_{X_i}(2t) = (1 - 2t)^{-1}, \ t < \frac{1}{2},$$

which means that $Y_i \sim \Gamma(1, 2)$, or $Y_i \sim \chi^2(2)$. Hence,

$$\frac{X_1}{X_2} = \frac{Y_1/2}{Y_2/2} \sim F(2, 2).$$