Exercises in Introduction to Mathematical Statistics (Ch. 7)

Tomoki Okuno

September 16, 2022

Note

- Not all solutions are provided: Exercises that are too simple or not very important to me are skipped.
- Texts in red are just attentions to me. Please ignore them.

7 Sufficiency

7.1 Measures of Quality of Estimators

Note that loss function problems were skipped because this kind of topic was not covered in my class.

7.1.1. Show that the mean $\overline{X}$ of a random sample of size n from a distribution having pdf $f(x;\theta) = (1/\theta)e^{-(x/\theta)}$, $0 < x < \infty$, $0 < \theta < \infty$, zero elsewhere, is an unbiased estimator of θ and has variance θ^2/n.

Solution. Since $X \sim \Gamma(1, \theta)$, $E(X) = \theta$ and $\text{Var}(X) = \theta^2$. Thus, $E(\overline{X}) = \theta$ and $\text{Var}(\overline{X}) = \theta^2/n$.

7.1.2. Let $X_1, X_2, ..., X_n$ denote a random sample from a normal distribution with mean zero and variance θ, $0 < \theta < \infty$. Show that $\sum_1^n X_i^2/n$ is an unbiased estimator of θ and has variance $2\theta^2/n$.

Solution.

Since $X/\sqrt{\theta}$ are iid $N(0,1)$, $\sum_i X_i^2/\theta \sim \chi^2(n)$. Hence,

$$E\left(\sum X_i^2/\theta\right) = n, \ \Rightarrow \ E\left(\sum X_i/n\right) = \theta,$$
$$\text{Var}\left(\sum X_i^2/\theta\right) = 2n, \ \Rightarrow \ \text{Var}\left(\sum X_i/n\right) = 2\theta^2/n.$$

7.1.3. Let $Y_1 < Y_2 < Y_3$ be the order statistics of a random sample of size 3 from the uniform distribution having pdf $f(x;\theta) = 1/\theta$, $0 < x < \theta$, $0 < \theta < \infty$, zero elsewhere. Show that $4Y_1$, $2Y_2$, and $\frac{4}{3}Y_3$ are all unbiased estimators of θ. Find the variance of each of these unbiased estimators.

Solution.

Note that the order statistics from a uniform distribution have a beta distribution.

By the theorem of a pdf of the order statistic, we obtain

$$f_{Y_1}(y) = \frac{3!}{0!2!}[1 - F_X(y)]^2 f_X(y) = \frac{3}{\theta}\left(1 - \frac{y}{\theta}\right)^2,$$
$$f_{Y_2}(y) = \frac{3!}{1!1!}F_X(y)[1 - F_X(y)]f_X(y) = \frac{6}{\theta}\left(\frac{y}{\theta}\right)\left(1 - \frac{y}{\theta}\right),$$
$$f_{Y_3}(y) = \frac{3!}{2!0!}F_X(y)^2 f_X(y) = \frac{3}{\theta}\left(\frac{y}{\theta}\right)^2.$$

Hence, let $y/\theta = w$, $dy = \theta dw$, then

$$E(Y_1) = \int_0^\theta 3\frac{y}{\theta}\left(1-\frac{y}{\theta}\right)^2 dy = 3\theta\int_0^1 w\,(1-w)^2\,dw = 3\theta\frac{\Gamma(2)\Gamma(3)}{\Gamma(5)} = \frac{\theta}{4},$$
$$E(Y_2) = \int_0^\theta 6\left(\frac{y}{\theta}\right)^2\left(1-\frac{y}{\theta}\right) dy = 6\theta\int_0^1 w^2\,(1-w)\,dw = 6\theta\frac{\Gamma(3)\Gamma(2)}{\Gamma(5)} = \frac{\theta}{2},$$
$$E(Y_3) = \int_0^\theta 3\left(\frac{y}{\theta}\right)^3 dy = 3\theta\int_0^1 w^3 dw = 3\theta\frac{\Gamma(4)\Gamma(1)}{\Gamma(5)} = \frac{3\theta}{4},$$

which is the desired result.

7.1.4. Let Y_1 and Y_2 be two independent unbiased estimators of θ. Assume that the variance of Y_1 is twice the variance of Y_2. Find the constants k_1 and k_2 so that $k_1Y_1 + k_2Y_2$ is an unbiased estimator with the smallest possible variance for such a linear combination.

Solution.

Given that $k_1Y_1 + k_2Y_2$ is unbiased,

$$E(k_1Y_1 + k_2Y_2) = (k_1+k_2)\theta \;\Rightarrow\; k_1 + k_2 = 1.$$

Hence,

$$\begin{aligned}\mathrm{Var}(k_1Y_1 + k_2Y_2) &= (k_1^2 + k_2^2/2)\mathrm{Var}Y_1\\ &= [2k_1 + (1-k_1)^2]\mathrm{Var}Y_1/2\\ &= (3k_1^2 - 2k_1 + 1)\mathrm{Var}Y_1/2\\ &= [3(k_1-1/3)^2 + 2/3]\mathrm{Var}Y_1/2\\ &\ge (1/3)\mathrm{Var}Y_1,\end{aligned}$$

suggesting that $k_1 = 1/3, k_2 = 2/3$ that minimize the variance for $k_1Y_1 + k_2Y_2$.

7.2 A Sufficient Statistic for a Parameter

Here, I used the definition of the exponential family as appropriate: Suppose

$$f(x;\theta) = h(x)k(\theta)e^{T(x)c(\theta)},$$

where $c(\theta)$ is nonconstant, $T'(x)$ is continuous. Then $T = \sum T(X_i)$ is (complete) and sufficient for θ.

7.2.1. Let $X_1, X_2, ..., X_n$ be iid $N(0,\theta)$, $0<\theta<\infty$. Show that $\sum_1^n X_i^2$ is a sufficient statistic for θ.

Solution.

The pdf of X is $f(x;\theta) = (2\pi\theta)^{-1/2}e^{-x^2/(2\theta)}$, which is clearly a member of the exponential family where $T(x) = x^2$. Hence, $T = \sum_1^n T(X_i) = \sum_1^n X_i^2$ is sufficient for θ.

7.2.2. Prove that the sum of the observations of a random sample of size n from a Poisson distribution having parameter θ, $0<\theta<\infty$, is a sufficient statistic for θ.

Solution.

The pdf of X is $f(x;\theta) = (x!)^{-1}e^{-\theta}e^{x\log\theta}$, which is a member of the exponential family where $T(x) = x$. Hence, $T = \sum_1^n T(X_i) = \sum_1^n X_i$ is a sufficient statistic for θ.

7.2.3. Show that the nth order statistic of a random sample of size n from the uniform distribution having pdf $f(x;\theta) = 1/\theta$, $0<x<\theta$, $0<\theta<\infty$, zero elsewhere, is a sufficient statistic for θ. Generalize this result by considering the pdf $f(x;\theta) = Q(\theta)M(x)$, $0<x<\theta$, $0<\theta<\infty$, zero elsewhere. Here, of course,

$$\int_0^\theta M(x)dx = \frac{1}{Q(\theta)}.$$

Solution.

Show only the general case. The joint pdf, or likelihood function, is given by

$$\begin{aligned} L(\theta;\mathbf{x}) &= [Q(\theta)]^n \prod_1^n M(x_i) I(0 < x_i < \theta) \\ &= [Q(\theta)]^n I(0 < y_n < \theta) \prod_1^n M(x_i) \\ &\equiv k(\mathbf{x};\theta)h(\mathbf{x}), \end{aligned}$$

zero elsewhere. By the factorization theorem, Y_n is a sufficient statistic for θ.

7.2.4. Let $X_1, X_2, ..., X_n$ be a random sample of size n from a geometric distribution that has pmf $f(x;\theta) = (1-\theta)^x\theta$, $x = 0, 1, 2, ...$, $0 < \theta < 1$, zero elsewhere. Show that $\sum_1^n X_i$ is a sufficient statistic for θ.

Solution.

The pdf of X (Geomatric distribution) is expressed as $f(x;\theta) = \theta e^{x\log(1-\theta)}$, which is a member of the exponential family where $T(x) = x$. Hence, $T = \sum_1^n T(X_i) = \sum_1^n X_i$ is a sufficient statistic for θ.

7.2.5. Show that the sum of the observations of a random sample of size n from a gamma distribution that has pdf $f(x;\theta) = (1/\theta)e^{-x/\theta}$, $0 < x < \infty$, $0 < \theta < \infty$, zero elsewhere, is a sufficient statistic for θ.

Solution.

The pdf of X clearly shows that $\Gamma(1,\theta)$ or Exponential distribution is a member of the exponential family where $T(x) = x$. Hence, $T = \sum_1^n T(X_i) = \sum_1^n X_i$ is a sufficient statistic for θ, which is the desired result.

7.2.6. Let $X_1, X_2, ..., X_n$ be a random sample of size n from a beta distribution with parameters $\alpha = \theta$ and $\beta = 5$. Show that the product $X_1X_2\cdots X_n$ is a sufficient statistic for θ.

Solution.

The pdf of X is expressed as $f(x;\theta) = B(\theta,5)^{-1}(1-x)^4 e^{(\theta-1)\log x}$, which is a member of the exponential family where $T(x) = \log x$. Hence, $T = \sum_1^n \log X_i = \log\prod_1^n X_i$, i.e., $\prod_1^n X_i$ is a sufficient statistic for θ.

7.2.7. Show that the product of the sample observations is a sufficient statistic for $\theta > 0$ if the random sample is taken from a gamma distribution with parameters $\alpha = \theta$ and $\beta = 6$.

Solution.

The pdf of $\Gamma(\theta, 6)$ is expressed as $f(x;\theta) = (\Gamma(\theta)6^\theta)^{-1}e^{-x/6}e^{(\theta-1)\log x}$, which is a member of the exponential family where $T(x) = \log x$. Hence, $T = \sum_1^n \log X_i = \log\prod_1^n X_i$, i.e., $\prod_1^n X_i$ is a sufficient statistic for θ.

7.2.8. What is the sufficient statistic for θ if the sample arises from a beta distribution in which $\alpha = \beta = \theta > 0$?

Solution.

The pdf of Beta(θ,θ) is given by $f(x,\theta) = B(\theta,\theta)^{-1}\exp\left[(\theta-1)\log x(1-x)\right]$, which is a member of the exponential family because $h(x) = 1, k(\theta) = B(\theta,\theta)^{-1}, T(x) = \log x(1-x)$ and $c(\theta) = \theta - 1$. Hence, $\sum_1^n \log X_i(1-X_i)$ or $\prod_1^n X_i(1-X_i)$ is sufficient for θ.

7.3. Properties of a Sufficient Statistic

7.3.1. In each of Exercises 7.2.1–7.2.4, show that the mle of θ is a function of the sufficient statistic for θ.

Solution.

The mles of Exercises 7.2.1–7.2.4 are, respectively, $n^{-1}\sum X_i^2$, $\overline{X}$, Y_n, and $(1+\overline{X})^{-1}$, which are a function of each sufficient statistic for θ.

7.3.2. Let $Y_1 < Y_2 < Y_3 < Y_4 < Y_5$ be the order statistics of a random sample of size 5 from the uniform distribution having pdf $f(x;\theta) = 1/\theta$, $0 < x < \theta$, $0 < \theta < \infty$, zero elsewhere. Show that $2Y_3$ is an unbiased estimator of θ. Determine the joint pdf of Y_3 and the sufficient statistic Y_5 for θ. Find the conditional expectation $E(2Y_3|y_5) = \phi(y_5)$. Compare the variances of $2Y_3$ and $\phi(Y_5)$.

Solution.

$$f_{Y_3}(y_3) = \frac{5!}{2!2!}F_X(y_3)^2[1 - F_X(y_3]^2 f_X(y_3) = \frac{30}{\theta}\left(\frac{y_3}{\theta}\right)^2\left(1 - \frac{y_3}{\theta}\right)^2.$$

Let $y/\theta = w$, then

$$E(Y_3) = \int_0^1 30\theta w^3(1-w)^2 dw = 30\theta\frac{\Gamma(4)\Gamma(3)}{\Gamma(7)} = \frac{\theta}{2},$$

indicating $2Y_3$ is unbiased for θ.

The pdf of Y_5 and the joint pdf of Y_3 and Y_5 are, respectively,

$$f_{Y_5}(y_5) = 5\frac{y_5^4}{\theta_5} \quad 0 < y_5 < \theta$$

$$f_{Y_3,Y_5}(y_3, y_5) = \cdots = \frac{60}{\theta^2}\left(\frac{y_3}{\theta}\right)^2\left(\frac{y_5 - y_3}{\theta}\right) = \frac{60}{\theta^5}y_3^2(y_5 - y_3), \quad 0 < y_3 < y_5 < \theta.$$

Hence,

$$E(2Y_3|y_5) = \int_0^{y_5} 2y_3\frac{60y_3^2(y_5 - y_3)/\theta^5}{5y_5^4/\theta_5} = \frac{24}{y_5}\int_0^{y_5}(y_3^3 y_5 - y_3^4)dy_3 = \frac{6y_5}{5}.$$

Since $E(Y_3^2) = 2\theta^2/7$, $\text{Var}(Y_3) = 2\theta^2/7 - (\theta/2)^2 = \theta^2/28$. Hence, $\text{Var}(2Y_3) = 4\text{Var}(Y_3) = \theta^2/7$. Also,

$$E(Y_5) = \cdots = \frac{5}{6}\theta,\ E(Y_5^2) = \cdots = \frac{5}{7}\theta^2 \ \Rightarrow\ \text{Var}(Y_5) = \frac{5}{7}\theta^2 - \frac{25}{36}\theta^2 = \frac{5}{(36)(7)}\theta^2.$$

Therefore,

$$\text{Var}(\phi(Y_5)) = \text{Var}(6Y_3/5) = \frac{36}{25}\text{Var}(Y_5) = \frac{1}{35}\theta^2 < \text{Var}(2Y_3),$$

which is the desired result.

7.3.3. If X_1, X_2 is a random sample of size 2 from a distribution having pdf $f(x;\theta) = (1/\theta)e^{-x/\theta}$, $0 < x < \infty$, $0 < \theta < \infty$, zero elsewhere, find the joint pdf of the sufficient statistic $Y_1 = X_1 + X_2$ for θ and $Y_2 = X_2$. Show that Y_2 is an unbiased estimator of θ with variance θ^2. Find $E(Y_2|y_1) = \phi(y_1)$ and the variance of $\phi(Y_1)$.

Solution.

First, the joint pdf of X_1 and X_2 is

$$f_{X_1,X_2}(x_1, x_2) = \theta^{-2}e^{-(x_1+x_2)/\theta},\ 0 < x_1 < \infty,\ 0 < x_2 < \infty.$$

The inverse functions are $x_1 = y_1 - y_2$ and $x_2 = y_2$, which gives us $J = 1$. So, the joint pdf of Y_1 and Y_2 is

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(y_1 - y_2, y_2)|J| = \theta^{-2}e^{-y_1/\theta},\ 0 < y_2 < y_1 < \infty.$$

Since $Y_2 = X_2 \sim \Gamma(1, \theta)$, $E(Y_2) = \theta$ and $\text{Var}(Y_2) = \theta^2$.

Next, the pdf of Y_1 is

$$f_{Y_1}(y_1) = \int_0^{y_1}\theta^{-2}e^{-y_1/\theta}dy_2 = \theta^{-2}y_1 e^{-y_1/\theta},$$

which gives the conditonal pdf:

$$f_{Y_2|Y_1}(y_2|y_1) = \frac{f_{Y_1,Y_2}(y_1,y_2)}{f_{Y_1}(y_1)} = y_1^{-1},\ 0 < y_2 < y_1 < \infty.$$

Hence,

$$E(Y_2|y_1) = \int_0^{y_1} y_2 f_{Y_2|Y_1}(y_2|y_1)dy_2 = y_1^{-1}\int_0^{y_1} y_2 dy_2 = \frac{y_1}{2}.$$

Since $Y_1 \sim \Gamma(2,\theta)$, $\mathrm{Var}(Y_1) = 2\theta^2$. So, $\mathrm{Var}(\phi(Y_1)) = \mathrm{Var}(Y_1)/4 = \theta^2/2$.

7.3.4. Let $f(x,y) = (2/\theta^2)e^{-(x+y)/\theta}$, $0 < x < y < \infty$, zero elsewhere, be the joint pdf of the random variables X and Y.

(a) Show that the mean and the variance of Y are, respectively, $3\theta/2$ and $5\theta^2/4$.

Solution.

$$f_Y(y) = \int_0^y (2/\theta^2)e^{-(x+y)/\theta}dx = (2/\theta)(e^{-y/\theta} - e^{-2y/\theta}).$$

Since the first and the second term follow $2\Gamma(1,\theta)$ and $\Gamma(1,\theta/2)$, respectively, $E(Y) = 2\theta - \theta/2 = 3\theta/2$. Also, $E(Y^2) = \cdots = 7\theta^2/2$ indicating that $\mathrm{Var}(Y) = 7\theta^2/2 - (3\theta/2)^2 = 5\theta^2/4$.

(b) Show that $E(Y|x) = x + \theta$. In accordance with the theory, the expected value of $X + \theta$ is that of Y, namely, $3\theta/2$, and the variance of $X + \theta$ is less than that of Y. Show that the variance of $X + \theta$ is in fact $\theta^2/4$.

Solution.

Since $f_X(x) = \int_x^\infty (2/\theta^2)e^{-(x+y)/\theta}dy = (2/\theta)e^{-2x/\theta}$,

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = (1/\theta)e^{(x-y)/\theta},$$
$$E(Y|X=x) = \int_x^\infty y(1/\theta)e^{(x-y)/\theta}dy = \cdots = x + \theta,$$

implies that $E(Y|X) = X + \theta$. And $E_X(E(Y|X)) = E(Y) = 3\theta/2$ by iterative expectation.

Since $X \sim \Gamma(1,\theta/2)$, $\mathrm{Var}(X+\theta) = \mathrm{Var}(X) = \theta^2/4$.

7.3.5. In each of Exercises 7.2.1–7.2.3, compute the expected value of the given sufficient statistic and, in each case, determine an unbiased estimator of θ that is a function of that sufficient statistic alone.

Solution.

For 7.2.1, $E(\sum_1^n X_i^2) = nE(X^2) = n\theta$ indicate that $\sum_1^n X_i^2/n$ is an unbiased estimator of θ.

For 7.2.2, $E(\sum_1^n X_i) = nE(X) = n\theta$ indicate that $\sum_1^n X_i/n$ is an unbiased estimator of θ.

For 7.2.3, $f_{Y_n}(y) = ny^{n-1}/\theta^n$ and $E(Y_n) = \frac{n}{n+1}\theta$ indicate that $\frac{n+1}{n}Y_n$ is an unbiased estimator of θ.

7.3.6. Let $X_1, X_2, ..., X_n$ be a random sample from a Poisson distribution with mean θ. Find the conditional expectation $E(X_1 + 2X_2 + 3X_3|\sum_1^n X_i)$.

Solution.

First, the expectation can be expanded by:

$$\begin{aligned}E(X_1 + 2X_2 + 3X_3|\sum_1^n X_i) &= E(X_1|\sum_1^n X_i) + 2E(X_2|\sum_1^n X_i) + 3E(X_3|\sum_1^n X_i)\\ &= 6E(X_1|\sum_1^n X_i) \quad \text{since } X_i \text{ are iid}\end{aligned}$$

The conditional probability

$$P(X_1 = x_1 | \sum_1^n X_i = x) = \frac{P(X_1 = x)P(\sum_2^n X_i = x - x_1)}{P(\sum_1^n X_i = x)} = \cdots = \binom{x}{x_1}\left(\frac{1}{n}\right)^{x_1}\left(1 - \frac{1}{n}\right)^{x-x_1},$$

indicates that

$$X_1 = x_1 \Big| \sum_1^n X_i = x \sim \text{Binomial}\left(x, \frac{1}{n}\right).$$

Hence, the expectation is

$$E(X_1 | \sum_1^n X_i = x) = \frac{x}{n} \;\Rightarrow\; E(X_1 | \sum_1^n X_i) = \frac{\sum_1^n X_i}{n} = \overline{X},$$

which gives us

$$E(X_1 + 2X_2 + 3X_3 \Big| \sum_1^n X_i) = 6\overline{X}.$$

7.4. Completeness and Uniqueness

7.4.1. If $az^2 + bz + c = 0$ for more than two values of z, then $a = b = c = 0$. Use this result to show that the family $\{b(2, \theta) : 0 < \theta < 1\}$ is complete.

Solution.

Suppose $E[g(X)] = 0$, then

$$\begin{aligned}\sum_{x=0}^{2} g(x)\binom{2}{x}\theta^x(1-\theta)^{2-x} &= g(0)(1 - 2\theta + \theta^2) + 2g(1)(\theta - \theta^2) + g(2)\theta^2 \\ &= [g(0) - 2g(1) + g(2)]\theta^2 + [-2g(0) + 2g(1)]\theta + g(0) \\ &= 0\end{aligned}$$

requires $g(0) - 2g(1) + g(2) = -2g(0) + 2g(1) = g(0)$, i.e., $g(0) = g(1) = g(2) = 0$, which is the desired result.

7.4.2. Show that each of the following families is not complete by finding at least one nonzero function $u(x)$ such that $E[u(X)] = 0$, for all $\theta > 0$.

(a)

$$f(x;\theta) = \begin{cases} \frac{1}{2\theta} & -\theta < x < \theta \\ 0 & \text{elsewhere.} \end{cases}$$

Solution. Since $X \sim U(-\theta, \theta)$, $E(X) = 0$. Thus, $u(x) = x$ is one nonzero function we want.

(b) $N(0, \theta)$, where $0 < \theta < \infty$.

Solution. We know $E(X) = 0$. Thus, $u(x) = x$ is one nonzero function that is desired.

7.4.3. Let $X_1, X_2, ..., X_n$ represent a random sample from the discrete distribution having the pmf

$$f(x;\theta) = \begin{cases} \theta^x(1-\theta)^{1-x} & x = 0, 1,\; 0 < \theta < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Show that $Y_1 = \sum_1^n X_i$ is a complete sufficient statistic for θ. Find the unique function of Y_1 that is the MVUE of θ.

Solution.

We know that X has a Bernoulli distribution that is a member of the exponential family. Then we can say $Y_1 \sim \text{Binom}(n, \theta)$ is a complete sufficient statistic for θ. Since $E(Y_1) = n\theta$, Y_1/n is the MVUE of θ.

7.4.4 Consider the family of probability density functions $\{h(z;\theta) : \theta \in \Omega\}$, where $h(z;\theta) = 1/\theta$, $0 < z < \theta$, zero elsewhere.

(a) Show that the family is complete provided that $\Omega = \{\theta : 0 < \theta < \infty\}$.
Hint: For convenience, assume that $u(z)$ is continuous and note that the derivative of $E[u(Z)]$ with respect to θ is equal to zero also.

Solution.

This is a simple case. Suppose $E[u(Z)] = 0$, then

$$\int_0^\theta u(z)/\theta dz = 0 \;\Rightarrow\; \frac{d}{d\theta}\int_0^\theta u(z)/\theta dz = 0 \;\Rightarrow\; u(\theta) = 0 \;(\theta > 0).$$

Since $z > 0$, it says $g(z) = 0$, which is the desired result.

(b) Show that this family is not complete if $\Omega = \{\theta : 1 < \theta < \infty\}$.
Hint: Concentrate on the interval $0 < z < 1$ and find a nonzero function $u(z)$ on that interval such that $E[u(Z)] = 0$ for all $\theta > 1$.

Solution.

This is a complicated case since $E[u(Z)] = 0 \Rightarrow u(\theta) = 0$, $\theta > 1$, which does not contain $0 < z < 1$. In this cases,

$$E[u(Z)] = 0 \;\Rightarrow\; \int_0^\theta u(z)/\theta dz = \int_0^1 u(z)/\theta dz + \int_1^\theta u(z)/\theta dz = 0.$$

Consider to make the first term on the left side zero. let

$$u(z) = \begin{cases} z - \frac{1}{2} & 0 < z < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Then, we find that

$$\int_0^1 (z - 1/2)/\theta dz + \int_1^\theta 0/\theta dz = \left[\frac{1}{2\theta}\left(z - \frac{1}{2}\right)^2\right]_0^1 = 0.$$

7.4.5. Show that the first order statistic Y_1 of a random sample of size n from the distribution having pdf $f(x;\theta) = e^{-(x-\theta)}$, $\theta < x < \infty$, $-\infty < \theta < \infty$, zero elsewhere, is a complete sufficient statistic for θ. Find the unique function of this statistic which is the MVUE of θ.

Solution.

$$L(\theta; \mathbf{x}) = \prod_1^n e^{-(x_i-\theta)} I(\theta < x_i) = e^{-(\sum x_i - n\theta)} I(\theta < y_1),$$

indicating that Y_1 is sufficient for θ. Then, the pdf of Y_1 is

$$f_{Y_1}(y_1) = \cdots = ne^{-n(y_1-\theta)},\; y_1 > \theta.$$

Further, suppose $E[g(Y_1)] = 0$, then

$$\int_\theta^\infty g(y_1) n e^{-n(y_1-\theta)} dy_1 = 0 \;\Rightarrow\; ng(\theta) = 0 \;\Rightarrow\; g(\theta) = 0,\; -\infty < \theta < \infty.$$

Thus, $g(y_1) = 0$, for all $y_1 > \theta$ indicating that Y_1 is a complete statistic for θ. Finally,

$$E(Y_1) = \int_\theta^\infty n y_1 e^{-n(y_1-\theta)} dy_1 = \cdots = \theta + \frac{1}{n},$$

implies that $Y_1 - 1/n$ is the MVUE of θ by the Lehmann-Scheffe.

7.4.7. Let X have the pdf $f_X(x;\theta) = 1/(2\theta)$, for $-\theta < x < \theta$, zero elsewhere, where $\theta > 0$.

(a) Is the statistic $Y = |X|$ a sufficient statistic for θ? Why?

Solution.

Yes; because the joint pdf of X (or likelihood) is

$$L(\theta;\mathbf{x}) = \prod_1^n (2\theta)^{-1} I(-\theta < x_i < \theta) = (2\theta)^{-n} \prod_{i=1}^n I(|x_i| < \theta),$$

implies that $Y = |X|$ a sufficient statistic by the factorization theorem.

(b) Let $f_Y(y;\theta)$ be the pdf of Y. Is the family $\{f_Y(y;\theta) : \theta > 0\}$ complete? Why?

Solution.

$$F_Y(y) = P(Y \le y) = P(|X| \le y) = P(-y \le X \le y) = \begin{cases} 0 & y \le 0 \\ y/\theta & 0 < y < \theta \\ 1 & y \ge \theta, \end{cases}$$

gives us $f_Y(y) = 1/\theta$, $0 < \theta < 1$, zero elsewhere. i.e., $Y \sim U(0,\theta)$. Suppose $E[g(Y)] = 0$, then

$$\int_0^\theta g(y)/\theta dy = 0 \;\Rightarrow\; g(\theta)/\theta = 0,\; \theta > 0 \;\Rightarrow\; g(y) = 0,\; y > 0.$$

Hence, the answer is yes; Y is a complete statistic for θ.

7.4.9. Let $X_1, ..., X_n$ be iid with pdf $f(x;\theta) = 1/(3\theta)$, $-\theta < x < 2\theta$, zero elsewhere, where $\theta > 0$.

(a) Find the mle $\widehat{\theta}$ of θ.

Solution.

The joint pdf of X (or likelihood) is

$$\begin{aligned} L(\theta;\mathbf{x}) &= \prod_1^n (3\theta)^{-1} I(-\theta < x_i < 2\theta) \\ &= (3\theta)^{-n} I(-\theta < y_1 < y_n < 2\theta) \\ &= (3\theta)^{-n} I(\theta > -y_1 \text{ and } \theta > y_n/2), \end{aligned}$$

indicating that $\widehat{\theta} = \max(-Y_1, 0.5Y_n)$.

(b) Is $\widehat{\theta}$ a sufficient statistic for θ? Why?

Solution. Yes; by part (a) and the factorization theorem.

(c) Is $(n+1)\widehat{\theta}/n$ the unique MVUE of θ? Why?

Solution.

Skipped; the calculation should be so heavy. I will separate it into two cases: $\widehat{\theta} = -Y_1$ and $\widehat{\theta} = 0.5Y_n$ to show $E(\widehat{\theta}) = (n/(n+1)\theta$.

7.4.10. Let $Y_1 < Y_2 < \cdots < Y_n$ be the order statistics of a random sample of size n from a distribution with pdf $f(x;\theta) = 1/\theta$, $0 < x < \theta$, zero elsewhere. By Example 7.4.2, the statistic Y_n is a complete sufficient statistic for θ and it has pdf

$$g(y_n;\theta) = \frac{ny^{n-1}}{\theta^n},\ 0 < y_n < \theta,$$

and zero elsewhere.

(a) Find the distribution function $H_n(z;\theta)$ of $Z = n(\theta - Y_n)$.

Solution.

Since the cdf of Y_n is $G(y_n) = y^n/\theta^n,\ 0 < y_n < \theta$,

$$\begin{aligned} H_n(z;\theta) = P(Z \le z) = P(Y_n \ge \theta - z/n) &= 1 - P(Y_n < \theta - z/n) \\ &= 1 - G(\theta - z/n) \\ &= 1 - \frac{(\theta - z/n)^n}{\theta^n} \\ &= 1 - \left(1 - \frac{z/\theta}{n}\right)^n. \end{aligned}$$

(b) Find the $\lim_{n\to\infty} H_n(z;\theta)$ and thus the limiting distribution of Z.

Solution. By part (a), $H_n(z;\theta) \to 1 - e^{-z/\theta}$ as $n \to \infty$. That is $Z \sim \Gamma(1,\theta)$.

7.5. The Exponential Class of Distributions

7.5.1. Write the pdf

$$f(x;\theta) = \frac{1}{6\theta^4}x^3 e^{-x/\theta},\ 0 < x < \infty,\ 0 < \theta < \infty,$$

zero elsewhere, in the exponential form. If $X_1, X_2, ..., X_n$ is a random sample from this distribution, find a complete sufficient statistic Y_1 for θ and the unique function $\psi(Y_1)$ of this statistic that is the MVUE of θ. Is $\psi(Y_1)$ itself a complete sufficient statistic?

Solution.

$X \sim \Gamma(4,\theta)$ is clearly a member of the exponential family, so $Y_1 = \sum_1^n X_i$ is a complete sufficient statistic for θ. We know $Y_1 \sim \Gamma(4n,\theta)$ indicating $E(Y_1) = 4n\theta$. Thus, $Y_1/(4n) = \overline{X}/4$ is the MVUE of θ. Clearly, $\psi(Y_1)$, a function of Y_1 alone, is a complete sufficient statistic.

7.5.2. Let $X_1, X_2, ..., X_n$ denote a random sample of size $n > 1$ from a distribution with pdf $f(x;\theta) = \theta e^{-\theta x}$, $0 < x < \infty$, zero elsewhere, and $\theta > 0$. Then $Y = \sum_1^n X_i$ is a sufficient statistic for θ. Prove that $(n-1)/Y$ is the MVUE of θ.

Solution.

Since $X \sim \Gamma(1, 1/\theta)$, $Y \sim \Gamma(n, 1/\theta)$: $f_Y(y) = [\theta^n/\Gamma(n)]y^{n-1}e^{-\theta y}$, $0 < y < \infty$. Hence

$$E\left(\frac{1}{Y}\right) = \int_0^\infty \frac{\theta^n}{\Gamma(n)} y^{n-2} e^{-\theta y} dy = \frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-1)}{\theta^{n-1}} = \frac{\theta}{n-1},$$

indicating that $(n-1)/Y$ is the MVUE of θ.

7.5.3. Let $X_1, X_2, ..., X_n$ denote a random sample of size n from a distribution with pdf $f(x;\theta) = \theta x^{\theta-1}, 0 < x < 1$, zero elsewhere, and $\theta > 0$.

(a) Show that the geometric mean $(X_1X_2\cdots X_n)^{1/n}$ of the sample is a complete sufficient statistic for θ.

Solution.

$f(x;\theta)=\theta e^{(\theta-1)\log x}, 0<x<1$ implies that this distribution is a member of the exponential family. Since $\sum_1^n \log X_i = \log\prod_1^n X_i$, $\prod_1^n X_i$ is a complete sufficient statistic for θ. The statistic is one-to-one, so $(\prod_1^n X_i)^{1/n}$, the geometric mean, is also complete and sufficient for θ.

(b) Find the maximum likelihood estimator of θ, and observe that it is a function of this geometric mean.

Solution.

Solving $\ell'(\theta)=0$ and $\ell''(\theta)<0$, we obtain the mle, $\widehat{\theta}=-n/\log\prod_1^n X_i=-1/\log(\prod_1^n X_i)^{1/n}$, which is a function of this geometric mean.

7.5.6. Given that $f(x;\theta)=\exp[\theta K(x)+H(x)+q(\theta)]$, $a<x<b$, $\gamma<\theta<\delta$, represents a regular case of the exponential class, show that the moment-generating function $M(t)$ of $Y=K(X)$ is $M(t)=\exp[q(\theta)-q(\theta+t)]$, $\gamma<\theta+t<\delta$.

Solution.

$$\begin{aligned}M_Y(t)&=\int_a^b \exp[(\theta+t)K(x)+H(x)+q(\theta)]dx\\&=\exp[q(\theta)-q(\theta+t)]\int_a^b \exp[(\theta+t)K(x)+H(x)+q(\theta+t)]dx\\&=\exp[q(\theta)-q(\theta+t)]\int_a^b f(x;\theta+t)dx\\&=\exp[q(\theta)-q(\theta+t)],\quad \gamma<\theta+t<\delta.\end{aligned}$$

7.5.7. In the preceding exercise, given that $E(Y)=E[K(X)]=\theta$, prove that Y is $N(\theta,1)$.
Hint: Consider $M'(0)=\theta$ and solve the resulting differential equation.

Solution.

Let $\psi(t)=\log M(t)=q(\theta)-q(\theta+t)$. Then $\psi'(t)=-q'(\theta+t)$, so $E(Y)=\psi'(0)=-q'(\theta)$, indicating

$$-q'(\theta)=\theta \;\Rightarrow\; q(\theta)=-\theta^2/2+C \text{ (constant)}.$$

Hence,

$$\begin{aligned}M_Y(t)&=\exp[q(\theta)-q(\theta+t)]\\&=\exp[-\theta^2/2+C-(-(\theta+t)^2/2+C)]\\&=\exp[-\theta^2/2+(\theta+t)^2/2]\\&=\exp[\theta t+t^2/2]\end{aligned}$$

implies that $Y\sim N(\theta,1)$.

7.5.10. Let $X_1,X_2,...,X_n$ be a random sample from a distribution with pdf $f(x;\theta)=\theta^2xe^{-\theta x}$, $0<x<\infty$, where $\theta>0$.

(a) Argue that $Y=\sum_1^n X_i$ is a complete sufficient statistic for θ.

Solution.

$X\sim\Gamma(2,1/\theta)$ is a member of the exponential family with $T(X)=X$. Thus, $Y=\sum_1^n X_i$ is a complete sufficient statistic for θ.

(b) Compute $E(1/Y)$ and find the function of Y that is the unique MVUE of θ.

Solution.

Since we have $Y \sim \Gamma(2n, 1/\theta)$,

$$E(Y^{-1}) = \int_0^\infty \frac{\theta^{2n}}{\Gamma(2n)} y^{2n-2} e^{-\theta y} dy = \frac{\theta^{2n}}{\Gamma(2n)} \frac{\Gamma(2n-1)}{\theta^{2n-1}} = \frac{\theta}{2n-1}$$

indicating that $(2n-1)/Y$ is is the MVUE of θ.

7.5.11. Let $X_1, X_2, ..., X_n$, $n > 2$, be a random sample from the binomial distribution $b(1, \theta)$.

(a) Show that $Y_1 = X_1 + X_2 + \cdots + X_n$ is a complete sufficient statistic for θ.

Solution.

Since the Binomial distribution is a member of the exponential family and

$$f(x;\theta) = \theta^x (1-\theta)^{1-x} = e^{x\text{logit}(\theta)+\log(1-\theta)},\ x = 0, 1,$$

$Y_1 = \sum_1^n X_i$ is a complete sufficient statistic for θ.

(b) Find the function $\psi(Y_1)$ that is the MVUE of θ.

Solution. Since $Y_1 \sim b(n, \theta)$, $E(Y_1) = n\theta$. Thus, $\psi(Y_1) = Y_1/n$ is the MVUE of θ by part (a).

(c) Let $Y_2 = (X_1 + X_2)/2$ and compute $E(Y_2)$.

Solution. $2Y_2 = X_1 + X_2 \sim b(2, \theta)$ gives $E(2Y_2) = 2\theta \;\Rightarrow\; E(Y_2) = \theta$.

(d) Determine $E(Y_2|Y_1 = y_1)$.

Solution.

By the iterative expectation and part (c), $E_{Y_1}[E(Y_2|Y_1)] = E(Y_2) = \theta$. Thus, $E(Y_2|Y_1 = y_1)$ is MVUE of θ by the Rao-Blackwell and Lehmann-Scheffe theorems. By part (b), we found that $Y_1/n = \overline{X}$ is MVUE of θ, which shows $E(Y_2|Y_1 = y_1) = Y_1/n$.

7.5.12. Let $X_1, X_2, ..., X_n$ be a random sample from a distribution with pmf $p(x;\theta) = \theta^x(1-\theta), x = 0, 1, 2, \ldots,$ zero elsewhere, where $0 \le \theta \le 1$.

(a) Find the mle, $\widehat{\theta}$, of θ.

Solution.

Solving $\ell'(\theta) = 0$ and checking $\ell''(\theta) < 0$, we obtain

$$\widehat{\theta} = \frac{\overline{X}}{1+\overline{X}}.$$

(b) Show that $\sum_1^n X_i$ is a complete sufficient statistic for θ.

Solution. X is a member of the exponential family and $T(X) = X$, which implies the desired result.

(c) Determine the MVUE of θ.

Since X has a negative binomial with parameter 1 and $1-\theta$, a member of the exponential family,

$$E(\overline{X}) = E(X) = \frac{\theta}{1-\theta}$$

and thus

$$\overline{X} \text{ is MVUE of } \frac{\theta}{1-\theta}$$

$$\Rightarrow g(\overline{X}) \text{ is MVUE of } g\left(\frac{\theta}{1-\theta}\right)$$

Let $g(x) = \frac{x}{1+x}$, then

$$g(\overline{X}) = \frac{\overline{X}}{1+\overline{X}} = \widehat{\theta}, \quad g\left(\frac{\theta}{1-\theta}\right) = \theta.$$

Hence, the mle of θ is the MVUE of θ.

7.6. Functions of a Parameter

7.6.1. Let $X_1, X_2, ..., X_n$ denote a random sample from a distribution that is $N(\theta, 1), -\infty < \theta < \infty$. Find the MVUE of θ^2.

Solution.

$N(\theta, 1)$ is a member of the exponential family because

$$f(x;\theta) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} e^{-\frac{\theta^2}{2}} e^{x\theta} = h(x)k(\theta)e^{T(x)c(\theta)}.$$

Hence, $\sum_1^n X_i$ and $\overline{X}$ is a complete sufficient statistic for θ. Further, since

$$E(\overline{X}^2) = \text{Var}(\overline{X}) + E(\overline{X})^2 = \frac{1}{n} + \theta^2,$$

$\overline{X}^2 - 1/n$ is the MVUE of θ^2 as $f(\overline{X}) = \overline{X}^2 - 1/n$ is also a complete sufficient statistic.

7.6.2. Let $X_1, X_2, ..., X_n$ denote a random sample from a distribution that is $N(0, \theta)$. Then $Y = \sum X_i^2$ is a complete sufficient statistic for θ. Find the MVUE of θ^2.

Solution.

$N(0, \theta)$ is a member of the exponential family because

$$f(x;\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{x^2}{2\theta}} = k(\theta)e^{T(x)c(\theta)}.$$

Hence, $\sum_1^n X_i^2$ is a complete sufficient statistic for θ. Here, we know that $X_i/\sqrt{\theta}$ are iid $N(0,1)$ and then $\sum X_i^2/\theta = Y/\theta \sim \chi^2(n)$. Hence,

$$E(Y/\theta) = n \;\Rightarrow\; E(Y) = n\theta,$$
$$\text{Var}(Y/\theta) = 2n \;\Rightarrow\; \text{Var}(Y) = 2n\theta^2,$$

which follows $E(Y^2) = \text{Var}(Y) + E(Y)^2 = 2n\theta^2 + (n\theta)^2 = (n^2+2n)\theta^2$, indicating that $Y^2/(n^2+2n)$ is the MVUE of θ^2 because $Y^2/(n^2+2n)$, a function of the sufficient statistic Y, is also complete sufficient statistic.

7.6.6. Let $X_1, X_2, ..., X_n$ be a random sample from a Poisson distribution with parameter $\theta > 0$.

(a) Find the MVUE of $P(X \le 1) = (1+\theta)e^{-\theta}$.
Hint: Let $u(X_1) = 1$, $X_1 \le 1$, zero elsewhere, and find $E[u(X_1)|Y = y]$, where $Y = \sum_1^n X_i$.

Solution.

By Exercise 7.3.6, we have $X_1|Y = y \sim \text{Binomial}(y_1, 1/n)$. Hence,

$$\begin{aligned} E[u(X_1)|Y_1 = y_1] &= \sum_{x_1=0}^{1} \binom{y}{x_1}\left(\frac{1}{n}\right)^{x_1}\left(1-\frac{1}{n}\right)^{y-x_1} \\ &= \left(1-\frac{1}{n}\right)^{y} + \left(\frac{y}{n}\right)\left(1-\frac{1}{n}\right)^{y-1} \\ &= \left(\frac{n-1}{n}\right)^{y}\left(1+\frac{y}{n-1}\right), \end{aligned}$$

which gives us the MVUE of $(X \leq 1)$:

$$\left(\frac{n-1}{n}\right)^{Y}\left(1+\frac{Y}{n-1}\right).$$

(b) Express the MVUE as a function of the mle of θ.

Solution.

We know the mle of θ is $\widehat{\theta} = \overline{X} = Y/n$ in this case. Thus, we can express the MVUE as

$$\left(\frac{n-1}{n}\right)^{n\overline{X}}\left(1+\frac{n\overline{X}}{n-1}\right).$$

(c) Determine the asymptotic distribution of the mle of θ.

Solution. Since $E(X) = \text{Var}(X) = \theta$, CLT gives $\sqrt{n}(\overline{X}-\theta) \xrightarrow{D} N(0,\theta)$, that is, $\overline{X}$ approx. $N(\theta, \theta/n)$.

(d) Obtain the mle of $P(X \leq 1)$. Then use Theorem 5.2.9 to determine its asymptotic distribution.

Solution.

By the invariance of MLE, $\widehat{P(X \leq 1)} = (1+\widehat{\theta})e^{-\widehat{\theta}} = (1+\overline{X})e^{-\overline{X}}$. Let $g(x) = (1+x)e^{-x}$, which is continuous and $g'(x) = -xe^{-x}$. Then using the Delta method, we obtain

$$\begin{aligned} \sqrt{n}(g(\overline{X}) - g(\theta)) &\xrightarrow{D} N(0, [g'(\theta)]^2\theta) \\ \Rightarrow \sqrt{n}(\widehat{P(X \leq 1)} - (1+\theta)e^{-\theta}) &\xrightarrow{D} N(0, \theta^3 e^{-2\theta}) \\ \widehat{P(X \leq 1)} &\text{ approx. } N((1+\theta)e^{-\theta}, \theta^3 e^{-2\theta}/n). \end{aligned}$$

7.6.7 Let $X_1, X_2, ..., X_n$ denote a random sample from a Poisson distribution with parameter $\theta > 0$. From Remark 7.6.1, we know that $E[(-1)^{X_1}] = e^{-2\theta}$.

(a) Show that $E[(-1)^{X_1}|Y_1 = y_1] = (1-2/n)^{y_1}$, where $Y_1 = X_1 + X_2 + \cdots + X_n$.

Solution.

By Exercise 7.3.6, we have $X_1|Y_1 = y_1 \sim \text{Binomial}(y_1, 1/n)$. Hence,

$$\begin{aligned} E[(-1)^{X_1}|Y_1 = y_1] &= \sum_{x_1=0}^{n} (-1)^{x_1}\binom{y_1}{x_1}\left(\frac{1}{n}\right)^{x_1}\left(1-\frac{1}{n}\right)^{y_1-x_1} \\ &= \sum_{x_1=0}^{n}\binom{y_1}{x_1}\left(-\frac{1}{n}\right)^{x_1}\left(1-\frac{1}{n}\right)^{y_1-x_1} \\ &= \left(-\frac{1}{n}+1-\frac{1}{n}\right)^{y_1} \\ &= \left(1-\frac{2}{n}\right)^{y_1}, \end{aligned}$$

which implies that $(1-2/n)^{Y_1}$ is the MVUE of $e^{-2\theta}$.

(b) Show that the mle of $e^{-2\theta}$ is $e^{-2\overline{X}}$.

Solution. Since the mle of θ is $\widehat{\theta} = \overline{X}$ (omitted the proof), $\widehat{e^{-2\theta}} = e^{-2\overline{X}}$ by the invariance of MLE.

(c) Since $y_1 = n\overline{x}$, show that $(1-2/n)^{y_1}$ is approximately equal to $e^{-2\overline{x}}$ when n is large.

Solution.

$$\left(1-\frac{2}{n}\right)^{y_1} = \left(1-\frac{2}{n}\right)^{n\overline{x}} = \left[\left(1-\frac{2}{n}\right)^{n}\right]^{\overline{x}} \to e^{-2\overline{x}},$$

which follows that the MVUE of $e^{-2\theta}$ is identical with the MLE as $n \to \infty$.

7.6.10. Let $X_1, X_2, ..., X_n$ be a random sample with the common pdf $f(x) = \theta^{-1}e^{-x/\theta}$, for $x > 0$, zero elsewhere; that is, $f(x)$ is a $\Gamma(1,\theta)$ pdf.

(a) Show that the statistic $\overline{X} = n^{-1}\sum_{i=1}^{n} X_i$ is a complete and sufficient statistic for θ.

Solution.

Since $\Gamma(1,\theta)$ is a member of exponential family, $Y = \sum_{i=1}^{n} X_i$ is a complete and sufficient statistic for θ. Then, so does $\overline{X}$ because it is a function of Y.

(b) Determine the MVUE of θ.

Solution. $E(\overline{X}) = E(X) = \theta$. By part (a) and the Lehmann-Scheffe theorem, $\overline{X}$ is the MVUE of θ.

(c) Determine the mle of θ.

Solution. We know that $\widehat{\theta} = \overline{X}/\alpha = \overline{X}$ (omitted the proof).

(d) Often, though, this pdf is written as $f(x) = \tau e^{-\tau x}$, for $x > 0$, zero elsewhere. Thus $\tau = 1/\theta$. Use Theorem 6.1.2 to determine the mle of τ.

Solution. By the invariance of MLE, $\widehat{\tau} = 1/\widehat{\theta} = 1/\overline{X}$.

(e) Show that the statistic $\overline{X} = n^{-1}\sum_{i=1}^{n} X_i$ is a complete and sufficient statistic for τ. Show that $(n-1)/(n\overline{X})$ is the MVUE of $\tau = 1/\theta$. Hence, as usual, the reciprocal of the mle of θ is the mle of $1/\theta$, but, in this situation, the reciprocal of the MVUE of θ is not the MVUE of $1/\theta$.

Solution.

By the factorization theorem (and nature of the exponential family), $Y = \sum_{i=1}^{n} X_i$ is a complete and sufficient statistic for θ. Then, so does $\overline{X}$ because it is a function of Y. Also, we know $Y = n\overline{X} \sim \Gamma(n, 1/\theta)$ obtained from the mgf of X. Thus,

$$E\left(\frac{1}{Y}\right) = \frac{\tau^n}{\Gamma(n)}\frac{\Gamma(n-1)}{\tau^{n-1}} = \frac{\tau}{n-1},$$

indicating that $(n-1)/Y$ is the MVUE of $\tau = 1/\theta$.

(f) Compute the variances of each of the unbiased estimators in parts (b) and (e).

Solution.

For part (b),

$$\text{Var}(\overline{X}) = \frac{\text{Var}(X)}{n} = \frac{\theta^2}{n}.$$

For part (e),

$$E\left(\frac{1}{Y^2}\right) = \frac{\tau^n}{\Gamma(n)}\frac{\Gamma(n-2)}{\tau^{n-2}} = \frac{\tau^2}{(n-1)(n-2)}$$

$$\Rightarrow \text{Var}\left(\frac{1}{Y}\right) = E\left(\frac{1}{Y^2}\right) - E\left(\frac{1}{Y}\right)^2 = \frac{\tau^2}{(n-1)^2(n-2)}$$

$$\Rightarrow \text{Var}\left(\frac{n-1}{Y}\right) = \frac{\tau^2}{n-2}.$$

7.6.11. Consider the situation of the last exercise, but suppose we have the following two independent random samples: (1) $X_1, X_2, ..., X_n$ is a random sample with the common pdf $f_X(x) = \theta^{-1}e^{-x/\theta}$, for $x > 0$, zero elsewhere, and (2) $Y_1, Y_2, ..., Y_n$ is a random sample with common pdf $f_Y(y) = \theta e^{-\theta y}$, for $y > 0$, zero elsewhere. The last exercise suggests that, for some constant c, $Z = c\overline{X}/\overline{Y}$ might be an unbiased estimator of θ^2. Find this constant c and the variance of Z.

Solution.

We have $X \sim \Gamma(1, \theta)$ and $Y \sim \Gamma(1, 2/\theta)$. Hence,

$$\frac{2\sum_1^n X_i}{\theta} = \frac{2n\overline{X}}{\theta} \sim \Gamma(n, 2) = \chi^2(2n),$$
$$2\theta\sum_1^n Y_i = 2n\theta\overline{Y} \sim \Gamma(n, 2) = \chi^2(2n),$$

which gives us the F statistic:

$$F = \frac{(2n\overline{X}/\theta)/2n}{2n\theta\overline{Y}/2n} = \frac{\overline{X}}{\theta^2\overline{Y}} \sim F(2n, 2n).$$

Hence,

$$E(F) = E\left(\frac{\overline{X}}{\theta^2\overline{Y}}\right) = \frac{2n}{2n-2} = \frac{n}{n-1} \;\Rightarrow\; E\left(\frac{n-1}{n}\frac{\overline{X}}{\overline{Y}}\right) = \theta^2 \;\Rightarrow\; c = \frac{n-1}{n}$$
$$\mathrm{Var}(F) = \mathrm{Var}\left(\frac{\overline{X}}{\theta^2\overline{Y}}\right) = \frac{2(2n)^2(2n+2n-2)}{2n(2n-2)^2(2n-4)} = \frac{n(2n-1)}{(n-1)^2(n-2)}$$
$$\Rightarrow \mathrm{Var}(Z) = \left(\frac{n-1}{n}\right)^2 \mathrm{Var}\left(\frac{\overline{X}}{\overline{Y}}\right) = \left(\frac{n-1}{n}\right)^2 \frac{n(2n-1)}{(n-1)^2(n-2)}\theta^2 = \frac{2n-1}{n(n-2)}\theta^4.$$